Games and spans

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Introduction

Game semantics has been a very active domain in the last twenty years, achieving major results like providing fully abstract models, i.e. models in which denotational equivalence is the same as operational equivalence, of a certain number of languages including PCF (a form of simply-typed lambda-calculus with a fixed-point operator and case definition), see for example [HO00].
The fundamental idea behind game semantics is that the execution of a program is an interaction between the program and its environment, and thus can be seen as a game between those two entities, each answering to the other's moves. A little example may clarify that. Let us consider the function successor. Its semantics in game terms may be seen as

- The environment asks what is the output.
- To answer this question, the program needs the input, in answer to the environment move, it asks what is the input.
- The environment answers n.
- The player can now answer the first question by saying $n + 1$.

Thus the semantic of a program can be given by its strategy, i.e. how it answers to each possible environment move.
Lately, game semantics have taken a great many different forms, some trying to define a notion of concurrent games, like [AM99], or trying to get rid of alternation, as in [MM07], or to define a more linear notion of games to model linear logic, see [FP09]. Moreover, they can also be related to previous notions like affine sequential algorithms, see [Cur94]. But these are but some examples... and each new definition comes with its particularities and complications.
There is therefore a great need to propose a common theoretical ground on which to work, thus to unify and better comprehend all that has already been done. This has already been attempted in [CM10], and in a certain way in [HHM07].
One could argue that this is but one other presentation of game semantics among all that already exist, but we hope that the category of games that we define here is larger than what has been explored at thus encompasses all previous definitions, providing in particular a sense of concurrent games.
We will will proceed in five parts, the first is an introduction of the computation model used through out this report, event structures, as well as two alternating ways to describe them, and some lemmas on the constructions we will need. Then we will go on to describe a tool we will need later, and then give the principal example of its use, that is to construct the category of relations. Thirdly we will define the category that is the goal of all this, the category $\mathit{Span}_{\mathit{Inn}}$. Fourthly, we will desribe some monadic construct that were encountered during my internship and finally, we will show how the games of [HHM07], can be embedded in our category.

To finish this introduction, I would like to thank Glynn Winskel for offering me this internship, for the great time I had in Cambridge, for his help, always there when needed, and for his patience with my mistakes and silly ideas. I would also like to thank Pierre-Louis Curien and Sam Staton for their answers and our discussions.

Chapter 1

Event structures, prime algebraic domains and stable families

Most of what follows can be found in [Win07].

1.1 Event structures

Definition 1.1.1 (Event structure) **:**
An event structure is $(E, \mathfrak{C}, \leqslant)$ where E is a set, $\mathfrak{C} \subseteq \mathcal{P}_{\mathrm{f}}(E)$ and $\leqslant$ is a partial order on E such that

(i) For all $e \in E$, $\lceil e \rceil := \{e' \mid e' \leqslant e\}$ is finite.

(ii) For all $e \in E$, $\{e\} \in \mathfrak{C}$.

(iii) For all $Y \subseteq X$, $X \in \mathfrak{C}$ implies $Y \in \mathfrak{C}$.

(iv) For all $e \in X \in \mathfrak{C}$ and $e' \leqslant e$, $X \cup \{e'\} \in \mathfrak{C}$.

An event structure therefore describes a computation process where certain events need that certain other events have occurred to be able to occur (this causal dependency relation by the order) and certain events cannot occur if others have occurred (this is described by the consistence relation).
In certain cases the $\mathfrak{C}$ relation will determined solely by a conflict relation, i.e. a subset of E will be consistent if and only if it does not contain a pair in conflict. Moreover, for all $e \in E$, let us write $\lfloor e \rfloor := \{e' \mid e' < e\}$.
We will write $x \prec y$ for x is direct predecessor to y.

Definition 1.1.2 (Configurations) **:**
Let $(E, \mathfrak{C}, \leqslant)$ be an event structure, $X \subseteq E$ such that

(i) It is down-closed, i.e. $e' \leqslant e \in X$ implies $e' \in X$.

(ii) It is finitely consistent, i.e. for all $Y \subseteq_{\mathrm{f}} X$, $Y \in \mathfrak{C}$.

is a E-configuration.
We write $\mathcal{C}(E)$ for the set of E-configurations and $\mathcal{C}^o(E)$ for the set of finite E-configurations.

Definition 1.1.3 (Partial maps of event structures) **:**
A partial rigid map of event structures $f : E \rightharpoonup E'$ is a partial map from E to E' such that

(i) For all $X \in \mathcal{C}^o(E)$, $f(X) \in \mathcal{C}^o(E')$.

(ii) The map f is locally injective on configurations, i.e. for all $X \in \mathcal{C}^o(E)$, $f|_X$ is injective.

We say that it is rigid if for all e_1, $e_2 \in E$, if $f(e_1)\downarrow$ and $f(e_2)\downarrow$, then $e_1 \leqslant e_2$ implies $f(e_1) \leqslant f(e_2)$.

Event structures and partial maps form a category $\mathcal{E}_p$. Its subcategory consisting of all the rigid maps is called $\mathcal{E}_{pr}$.
The three following lemmas state some very basic, but useful properties of the maps of event structure.

Lemma 1.1.4 :

Let $f : E \rightharpoonup F$ be a partial map of event structures and $x, y \in E$ consistent such that $f(x) \leqslant f(y)$ (and both defined). Then $x \leqslant y$.

Δ. First, as $f(\lceil y \rceil)$ is down-closed, there exists $x' \in E$ such that $x' \leqslant y$ and $f(x') = f(x)$. But as x' is a predecessor to y and x and y are consistent, x' and x are consistent too and therefore must be equal. Thus $x \leqslant y$. □

Lemma 1.1.5 :

Let $f : E \rightharpoonup F$ be a partial map of event structures and $x, y \in E$ such that $x \prec y$ and $f(x) \leqslant f(y)$, then $f(x) \prec f(y)$.

Δ. If there exist $z \in F$ such that $f(x) \leqslant z \leqslant f(y)$ then there exist $t \leqslant y$ such that $f(t) = z$ and $x' \leqslant t$ such that $f(x') = f(x)$. But, as x and x' are both in the history of y, they must be consistent and thus equal. Therefore $x \leqslant t \leqslant y$ and t must be either x or y, which implies that z is either $f(x)$ or $f(y)$. □

Lemma 1.1.6 :

Let $f : E \rightharpoonup F$ be a rigid partial map of event structures and $x \in E$ such that $f(x)\downarrow$. Then $\lceil f(x) \rceil = f(\lceil x \rceil)$ and $\lfloor f(x) \rfloor = f(\lfloor x \rfloor)$.

Δ. $f(\lceil x \rceil) \in \mathcal{C}^o(F)$ and thus is down-closed. As $f(x) \in f(\lceil x \rceil)$, $\lceil f(x) \rceil \subseteq f(\lceil x \rceil)$. Now let us consider $y \in f(\lceil x \rceil)$. There exists $z \in \lceil x \rceil$ such that $f(z) = y$. But then, as $z \leqslant x$, $f(z) \leqslant f(x)$ and $y \in \lceil f(x) \rceil$. Moreover as $f|_{\lceil x \rceil}$ is injective, x is the then only element in $\lceil x \rceil$ to have $f(x)$ as an image and thus $f(\lfloor x \rfloor) = \lfloor f(x) \rfloor$.□

1.2 An equivalent presentation, prime algebraic domains

Let E be an event structure, the ordered set $(\mathcal{C}(E), subs)$ is in fact a domain with certain specificities that we describe here.

Definition 1.2.1 (Prime algebraic domains) **:**
Let $(D, \leqslant)$ be a partial order. Let $X \subseteq D$, we say that it is compatible if and only if it is bounded in D. We say that D is consistent complete if all X finitely compatible (i.e. all finite subsets are compatible) has a least upper bound $\coprod X$.
An element of D is a complete prime if and only if for all X finitely compatible, if $p \leqslant \coprod X$, then $p \leqslant x$ for some $x \in X$.
The partial order D is a prime algebraic domain if it is consistent complete and for all $d \in D$, $d = \coprod\{p \leqslant d \mid p$ is a complete prime$\}$.
It is finitary if all complete primes are superior to only a finite number of elements in D.

The following theorem, shows that this two presentations are indeed equivalent, it is proved in [NPW81].

Theorem 1.2.2 :

Let E be an event structure, then $(\mathcal{C}(E), \subseteq)$ is a finitary prime algebraic domain, whose prime are the histories.
Let $(D, \leqslant)$ be a finitary prime algebraic domain. Let P be the set of its prime, $\leqslant_P$ be $\leqslant$ and $X \subseteq_{\mathrm{f}} P$ be in $\mathfrak{C}_P$ if and only if X is compatible. Then $(P, \mathfrak{C}_P, \leqslant_P)$ is an event structure.
These two transformations are reciprocal, up to isomorphism.

1.3 Event structures as a co-reflective subcategory of stable families

For any event structure E, the set $\mathcal{C}^o(E)$ also has interesting properties, it is a stable family.

Definition 1.3.1 (Stable family) **:**
A stable family is a family of finite sets $\mathcal{F}$ such that

(i) It is complete, if $Z \subseteq_{\mathrm{f}} \mathcal{F}$ and Z is compatible, then $\bigcup Z \in \mathcal{F}$.

(ii) It is coincidence-free, for all $x \in \mathcal{F}$, e, $e' \in x$ such that $e \neq e'$, there exists $y \in \mathcal{F}$ such that $y \subseteq x$ and $e \in y$ if and only if $e \notin y$.

(iii) It is stable, for all $Z \subseteq \mathcal{F}$ compatible, $\bigcap Z \in \mathcal{F}$.

where compatibility is with respect to the inclusion order.

We call $\bigcup \mathcal{F}$ the set of events underlying the stable family.

Definition 1.3.2 (Partial maps of stable families) **:**
A partial map of stable family $f : \mathcal{E} \rightharpoonup \mathcal{F}$ is a partial map of sets $\bigcup \mathcal{E} \rightharpoonup \bigcup \mathcal{F}$, such that the image of an element of $\mathcal{E}$ is in $\mathcal{F}$ and that is injective on all elements of $\mathcal{E}$.

Stable families and partial maps form a category $\mathcal{F}_p$. Let us now show how we can go from an event structure to a stable family and vice versa.

Definition 1.3.3 ($\mathcal{C}^o()$) **:**
Let E be an event structure, then $\mathcal{C}^o(E)$ is a stable family. Let $f : E \rightharpoonup F$ be a partial map of event structures, then there is a partial map of stable families $\mathcal{C}^o(f) : \mathcal{C}^o(E) \rightharpoonup \mathcal{C}^o(F)$ that sends $X \in \mathcal{C}^o(E)$ to $f(X)$. Moreover, this makes $\mathcal{C}^o()$ a functor.

Definition 1.3.4 (Pr) **:**
Let $\mathcal{F}$ be a stable family. Let $x \in \mathcal{F}$ and $e \in x$. Let $\lceil e \rceil_x := \bigcap \{y \in \mathcal{F} \mid e \in y \subseteq x\}$. Let $P := \{\lceil e \rceil_x \mid e \in x \in \mathcal{F}\}$, it is ordered by inclusion, and let $X \in \mathfrak{C}_P$ if and only if $X \subseteq_{\mathrm{f}} P$ and $\bigcup X$. Then $(P, \mathfrak{C}_P, \leqslant_P)$ is an event structure.
Let $f : \mathcal{E} \rightharpoonup \mathcal{F}$ be a partial map of stable families, then $\mathrm{Pr}(f) : \mathrm{Pr}(\mathcal{E}) \rightharpoonup \mathrm{Pr}(mathcalF)$ that sends $\lceil e \rceil_x$ to $\lceil f(e) \rceil_{f(x)}$), when $f(e)\downarrow$ is a partial map of event structures. Moreover this makes Pr *a functor.*

Theorem 1.3.5 :
The functor $\mathcal{C}^o()$ is left adjoint to the functor Pr. The unit is an isomorphism, thus $\mathcal{E}_p$ is a co-reflective subcategory of $\mathcal{F}_p$.

1.4 Limits of event structures

The first utility we have of stable families is to define the limits of event structures (using the co-reflection), in particular the product. For that we first have to define the product of set with partial functions.

Definition 1.4.1 (Partial product) **:**
Let X and Y be two sets, the set

$$X \times_\star Y := ((X \sqcup \{\star\}) \times (Y \sqcup \{\star\})) \setminus \{(\star, \star)\}$$

is called the partial product of X and Y.

Let us also define the two following partial functions

$$\begin{array}{cccc} X \times_\star Y & \overset{\pi_X}{\to} & X & \\ (x, \_) & \mapsto & x & \text{if } x \neq \star \end{array}$$

$$\begin{array}{cccc} X \times_\star Y & \overset{\pi_Y}{\to} & Y & \\ (\_, y) & \mapsto & y & \text{if } y \neq \star \end{array}$$

They are called the projections.

Definition 1.4.2 (Product of stable families) **:**
Let $\mathcal{E}$ and $\mathcal{F}$ be two stable families. Let us write $\bigcup\mathcal{E} \times_\star \bigcup\mathcal{F}$ for the partial product of the underlying sets, with projections π_1 and π_2. Then $x \in \mathcal{E} \times \mathcal{F}$ if and only if

(i) $x \subseteq_{\mathrm{f}} \bigcup\mathcal{E} \times_\star \bigcup\mathcal{F}$.

(ii) $\pi_1(x) \in \mathcal{E}$ *and* $\pi_2(x) \in \mathcal{F}$.

(iii) For all e, $e' \in x$, *if* $\pi_i(e) = \pi_i(e')$ *for* $i = 1$ *or* 2, *then* $e = e'$.

(iv) For all e, $e' \in x$ *such that* $e \neq e'$, *there exists* $y \subseteq x$ *such that* $\pi_1(y) \in \mathcal{E}$, $\pi_2(y) \in \mathcal{F}$ *and* $e \in y$ *if and only if* $e' \notin y$.

Moreover, let $p_1 : \mathcal{E} \times \mathcal{F} \rightharpoonup \mathcal{E}$ *the extension to subsets of* π_1 *and* p_2 *that of* π_2.

The stable family $\mathcal{E} \times \mathcal{F}$ and the maps p_1, p_2 form the product in $\mathscr{F}_p$.

Theorem 1.4.3 (Product of event structures) **:**

Let E and F be two event structures then their product $E \times F$ in $\mathscr{E}_p$ is $\Pr(\mathcal{C}^o(E) \times \mathcal{C}^o(F))$. The left projection is the map that send an element of the form $\lceil (e, \_) \rceil_x$ to e and the right, the map that sends $\lceil (\_, e) \rceil_x$ to e.

Let us now prove some lemma about the product, that we will use extensively latter on. This first lemma simply states that in the product we cannot have events not related to any member of the product that appear.

Lemma 1.4.4 :

Let E and F be event structures, and let $x \in E \times F$, then at least one of $p_E(x)$ and $p_F(x)$ is defined.

Δ. Let x in $E \times F$, then there exists $y \in \mathcal{C}^o(E) \times fconf F$ and $e \in y$ such that $x = \lceil e \rceil_y$. As $p_E = \eta^{-1} \circ \pi_E$, $p_E(x) \doteq \pi_E(e)$, and identically for F. But $e \in E \times_\star F$ and thus at least one of $\pi_E(e)$ and $\pi_F(e)$ is defined.□

Then, this lemma states that order can only arise locally as inherited from one or the other member of the product.

Lemma 1.4.5 :

Let E and F be two event structures and let x, $y \in E \times F$ such that $x \prec y$ then there exist G among E and F such that $p_G(x)\downarrow$, and $p_G(y)\downarrow$ and $p_G(x) \prec p_G(y)$.

Δ. Let us suppose that $x \prec y$ but that $p_G(x)\downarrow$, and $p_G(y)\downarrow$ and $p_G(x) \prec p_G(y)$ is false in both cases.
By construction there exist s, $t \in \mathcal{C}^o(E) \times \mathcal{C}^o(F)$, $e \in s$ and $f \in t$ such that $x = \lceil e \rceil_s$ and $y = \lceil f \rceil_t$. But then, as $x \leqslant y$, $\lceil e \rceil_s \subseteq \lceil f \rceil_t$ and thus $e \in \lceil e \rceil_s \subseteq \lceil f \rceil_t \subseteq t$ ans therefore $\lceil e \rceil_t \subseteq \lceil e \rceil_s$. But that implies that $\in \lceil e \rceil_t \subseteq \lceil e \rceil_s \subseteq s$ and thus that $\lceil e \rceil_s \subseteq \lceil e \rceil_t$. We can therefore consider that $s = t$.
Let $z = y \setminus e$. Let us show that $t \in \mathcal{C}^o(E) \times \mathcal{C}^o(F)$. As $f \in z \subseteq t$, that would imply that $e \notin \lceil f \rceil_t$, but that is absurd.

Let us first show that $\pi_S(z) \in \mathcal{C}^o(S)$. By definition, $\pi_S(z) = \pi_S(\lceil f \rceil_t) \setminus \pi_S(e)$. If $\pi_S(e)\uparrow$, then it is done. Let us now suppose that $\pi_S(e)\downarrow$. As $\pi_S(z) \subseteq \pi_S(\lceil f \rceil_t)$, it is consistent. We therefore only need to show it is down-closed. A problem only arises if we have $\pi_S(e) \leqslant \pi_S(a)$ for some $a \in \lceil f \rceil_t$. But then as $\pi_S(\lceil a \rceil_t)$ is down-closed, there exists an $e' \in \lceil a \rceil_t \subseteq t$ such that $\pi_S(e') = \pi_S(e)$. But, as e, $e' \in t$ and have the same image by π_S, they must be equal. Thus $\lceil e \rceil_t \subseteq \lceil a \rceil_t \subseteq \lceil f \rceil_t$. But as $\lceil e \rceil_t \prec \lceil f \rceil_t$, that implies that $\lceil a \rceil_t = \lceil e \rceil_t$ or $\lceil a \rceil_t = \lceil f \rceil_t$. Because stable family are coincidence free, that implies that $a = e$ or $a = f$.
If $a = f$ then we have $\pi_S(e) \leqslant \pi_S(f)$. As we can't have $\pi_S(e) \prec \pi_S(f)$, it implies there is a $b \in \lceil f \rceil_t$ such that $\pi_S(e) < \pi_S(b) < \pi_S(f)$, but then there must be a $e' \in \lceil b \rceil_t$ such that $\pi_S(e) = \pi_S(e')$, but by local injectivity of π_S, $e = e'$ and thus $\lceil e \rceil_t \subset \lceil b \rceil_t \subset \lceil f \rceil_f$. But that contradicts $\lceil e \rceil_t \prec \lceil f \rceil_t$.
Thus $a = e$, and therefore $\pi_S(z)$ is down-closed. The same proof shows that $\pi_T(z) \in \mathcal{C}^o(T)$.
As $z \subset t$, it is clear that both π_S and π_T are injective on it.
Finally, let e', $e'' \in z \subseteq \lceil f \rceil_t$. Then by definition of the product, there exists $z' \subseteq \lceil f \rceil_t$ such that its projections are configurations and $e' \in z' \iff e'' \notin z'$. As e' and e'' are different from e, $z' \setminus e$ has the same separating property. Moreover $z' \setminus e \subseteq z$. Only remains to show that $\pi_S(z' \setminus e)$ is a configuration.
As $\pi_S(z' \setminus e) \subseteq \pi_S(z')$ it is consistent. The only problem if we try to show that it is down-closed is when we have $\pi_S(e) \leqslant \pi_S(a)$ for some a in z'. But as $z' \subseteq \lceil f \rceil_t$, we have already shown that that implies $a = e$ and thus $\pi_S(z')$ is down-closed. The same proof holds to show that $\pi_T(z') \in \mathcal{C}^o(T)$. Thus z is indeed an element of $\mathcal{C}^o(E) \times \mathcal{C}^o(F)$. □

Chapter 2

Transforming bicategories

2.1 The extremity of a bicategory

The construction explained in this chapter allows to take a bicategory and modify it following the action of an endofunctor on hom-categories. In fact what we do is define the composition to be the old composition followed by the endofunctor. This will be useful when defining a category of games transform the parallel composition to parallel composition plus hiding (the hiding being the endofunctor part).
Let us consider a bicategory $\mathcal{B}$ (with composition $\odot$ and identities $\imath$) and for all A, B 0-cells of $\mathcal{B}$, an endofunctor $\mathcal{E}_{A,B}$. We will suppose that we have a natural transformation $e^{A,B} : \mathrm{Id} \dot{\rightarrow} \mathcal{E}_{A,B}$ such that $\mathcal{E}(\mathrm{id} \odot e)$ (and symmetrically $\mathcal{E}(e \odot \mathrm{id})$) and $e_{\mathcal{E}}$ are isomorphisms.

Theorem 2.1.1 :

There is a bicategory $\mathcal{E}\,\mathcal{B}$ whose 0-cells are those of $\mathcal{B}$ and homcats are $\mathcal{E}(\mathcal{B}(A,B))$. The composition of a and b composable is $a \ominus b := \mathcal{E}(a \odot b)$, and the identities are $\jmath_A := \mathcal{E}(\imath_A)$.

Δ. We have first to construct the associator. Let α be the associator in $\mathcal{B}$, and let s, t and u be composable 1-cells in $\mathcal{E}\,\mathcal{B}$. Then we have

$$\begin{array}{ccc}
\mathcal{E}((s \odot t) \odot u) & \xrightarrow{\mathcal{E}\alpha} & \mathcal{E}(s \odot (t \odot u)) \\
\downarrow {\scriptstyle \mathcal{E}(e_{s \odot t} \odot \mathrm{id}_u)} & & \downarrow {\scriptstyle \mathcal{E}(\mathrm{id}_s \odot e_{t \odot u})} \\
(s \ominus t) \ominus u & \overset{}{\underset{\alpha'}{\dashrightarrow}} & s \ominus (t \ominus u)
\end{array}$$

But all these arrows are natural isomorphism, and so let us defined α' to be their composition.
As for the left unitor, let λ be the left unitor in $\mathcal{B}$, and $s : A \to B$ be a 1-cell in $\mathcal{E}\,\mathcal{B}$. Then we have

$$\begin{array}{ccc}
\mathcal{E}(\imath \odot s) & \xrightarrow{\mathcal{E}\lambda} & \mathcal{E}(s) \\
\downarrow {\scriptstyle \mathcal{E}(e_\imath \odot \mathrm{id}_s)} & & \uparrow {\scriptstyle e_{\mathcal{E}t}} \\
\jmath \ominus s & \underset{\lambda'}{\dashrightarrow} & s
\end{array}$$

where s is of the form $\mathcal{E}t$. Once more these three arrows are natural isomorphism and so let us define λ' to be their composition.
The construction of the right unitor, from the right unitor ρ of $\mathcal{B}$ follows symmetrically from this diagram

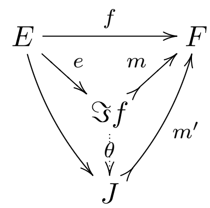
$$\begin{array}{ccc}
\mathcal{E}(s \odot \imath) & \xrightarrow{\mathcal{E}\rho} & \mathcal{E}(s) \\
\downarrow {\scriptstyle \mathcal{E}(\mathrm{id}_s \odot e_\imath)} & & \uparrow {\scriptstyle e_{\mathcal{E}t}} \\
s \ominus \jmath & \underset{\rho'}{\dashrightarrow} & s
\end{array}$$

The proof that the triangle and the pentagon commute can be found in figures (**A.1**) and (**A.2**) in the annexes. □

2.2 Relations

Let us now use this construction to construct the bicategory of relations from the bicategory of spans. It is given here mainly because it is this construction that inspired the "visible part" construction in the next chapter.
First of all, let us define the notion of image of a map. In $\mathcal{SET}$, this defines what we usual understand as the image.

Definition 2.2.1 (Image of a map) **:**
et A, B be objects of a category C and $f \in C(A,B)$ a map. The image of f is the smallest subobject of B through which f composes. In other terms, it is an object $\Im f$ of C, a monomorphism $m : \Im f \to B$ and a map $e : E \to \Im f$ such that $f = m \circ e$ and whenever we have



where m' is a monomorphism, there is a map θ (necessarily unique) such that the triangles commute.

Let us now introduce the notions of strong and extremal maps, that characterise the map from the domain to the image of a map, and prove some lemmas about them.

Definition 2.2.2 (Extremal map) **:**
An extremal map is a map such that whenever it factorises through a monomorphism then the monomorphism is an isomorphism.

Lemma 2.2.3 :
In a category with equalisers, an extremal map is an epimorphism.

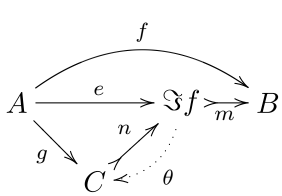
Δ. Let us consider $e : A \to B$ an extremal map and f, $g : B \to X$ such that $f \circ e = g \circ e$. The universal property of the equaliser e' of f and g is such that e must factorise through e'. But all equalisers are monomorphism and so e' must be an isomorphism, as e is extremal. As $f \circ e' = g \circ e'$, we have indeed $f = g$. □

Lemma 2.2.4 :
A monomorphism that is extremal is an isomorphism.

Δ. Let us consider $e : A \to B$ to be both a monomorphism and an extremal map. Then as $e = \mathrm{id} \circ e$, it factorises through the monomorphism e that must therefore be an isomorphism. □

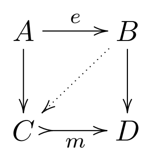
Lemma 2.2.5 :
Let $\mathcal{C}$ be a category and $f : A \to B$ an arrow of $\mathcal{C}$ that has an image $(e, \Im f, m)$, then e is an extremal map.

Δ. Let us consider the following diagram



where n is a monomorphism. Then $n \circ m$ is a monomorphism, through which f factorises. Thus, by the property of the image, there is map $\theta : \Im f \to C$ such that $m \circ n \circ \theta = m$ and $\theta \circ e = g$. But then, as m is a monomorphism, $n \circ \theta = \mathrm{id}$. Thus n is a monomorphism with a right inverse, thus an isomorphism. □

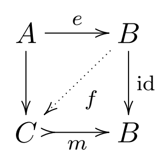
Definition 2.2.6 (Strong map) **:**
In a category $\mathcal{C}$, a map $e : A \to B$ is strong if and only if whenever there exists a commutative square



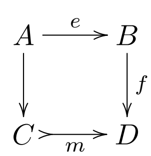
where m is a monomorphism, there exists a map $B \to C$ that cuts the square into two commutative triangles.

Lemma 2.2.7 :
In a category with pullbacks, a map is extremal if and only if it is strong.

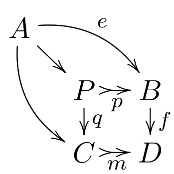
Δ. It is always he fact that a strong map is extremal. Indeed, let $e : A \to B$ be a strong map. If there exists a monomorphism m onto the codomain of e through which e factorises then we have the commutative square and thus the diagonal



Thus $m \circ f = \mathrm{id}$, and therefore m is a monomorphism and a split epimorphism, i.e. an isomorphism.
Let now e be an extremal map and let us consider the following commutative square



where m is a monomorphism. Then by taking the pullback P of m and f and by considering the map from A to P that exists by universal property, we obtain the following digram (bearing in mind that the pullback of a monomorphism is a monomorphism)



As e is an extremal map that factors through a monomorphism p, p must be an isomorphism, and the map $q \circ p^{-1}$ cuts the square into two commutative triangles. □

We now will now define the other central notion of this report, spans.

Definition 2.2.8 (Span) **:**
A span is a pair of maps with a common domain.

Spans are very useful in describing interaction between the co-domains of their two maps. In particular, they are the way we will represent strategies later on.
Let us now describe rapidly the categorical structure this spans have.

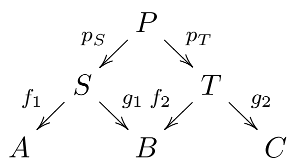
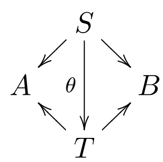


Figure 2.1: Span composition

Definition 2.2.9 (Span morphism) **:**
Let s and t be two spans in $\mathcal{E}_p$ with the same event structures at the feet. We say that a partial map $\theta : S \rightharpoonup T$ is a morphism of spans $\theta : s \Rightarrow t$ if



commutes.

Span morphisms compose like partial maps. Let A and B be two event structures, then the spans between A and B and the span morphisms form a category $\mathcal{Span}(A, B)$.
Until the end of this section, let $\mathcal{C}$ be a category with all pullbacks, binary products and images and such that the pullback of a strong map is still one. Then the spans in $\mathcal{C}$ form a bicategory $\mathcal{Span}_{\mathcal{C}}$ where composition is done according to figure **(2.1)** where P is the pullback. One may notice that the homcat $\mathcal{Span}_{\mathcal{C}}(A, B)$ where A and B are objets of $\mathcal{C}$ is in fact the category $\mathcal{C} \downarrow (A \times B)$.

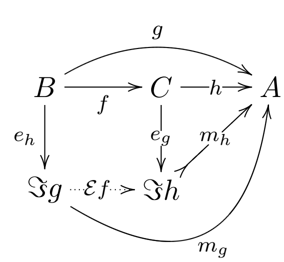
Definition 2.2.10 (Relation) **:**
A relation is a span that is jointly monic.

In $\mathcal{SET}$ a relation therefore correspond to an injection into the product of the two co-domains. That is exactly the usual understanding that we have of what is a relation.
Let us now defined the endofunctor on the hom-categories that will allow us to transform the category of spans into the category of relations.

Lemma 2.2.11 (The $\mathcal{E}$ functor) **:**

> Let $\mathcal{C}$ be a category with all images (and thus a choice of canonical images). Let A be an object of $\mathcal{C}$ There is a unique way to make $\mathcal{E}$ into a endofunctor on $\mathcal{C} \downarrow A$, that sends each object of the slice category $(B, f : B \to A)$ to its image $(\Im f, m_f)$, such that e is a natural transformation from $\mathrm{Id} \dot{\to} \mathcal{E}$.

Δ. Let us consider $f : (B, g) \to (C, h)$ in $\mathcal{C} \downarrow A$ and thus the following diagram



where g factorises through m_h a monomorphism and thus there exist a unique map (that we can call $\mathcal{E}f$) such that the diagram commutes. □

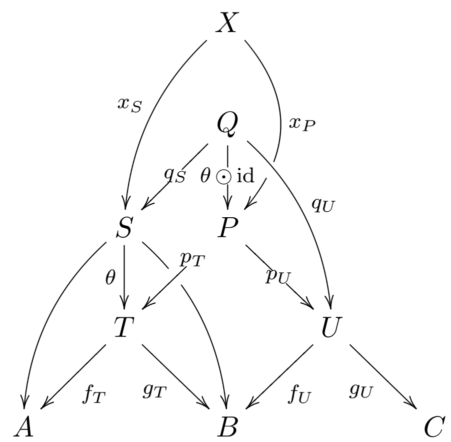


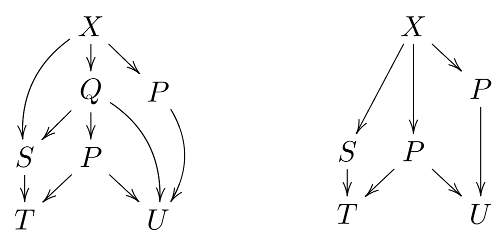
Figure 2.2: Preservation of strong maps

We now have to prove the two hypothesis that are used in the extremity construction. To prove the first we proceed in two stages. First we prove that $e \odot \mathrm{id}$ is strong and then that $\mathcal{E}$ of a strong map is an isomorphism.

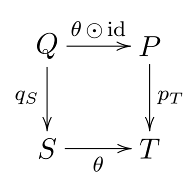
Lemma 2.2.12 :

Let $\odot$ denote the composition in $\mathcal{S}pan_{\mathcal{C}}$ and let θ be a span morphism that is also an extremal map in $\mathcal{C}$. Then $\mathrm{id} \odot \theta$ and $\theta \odot \mathrm{id}$ are strong.

Δ. Let us thus show that $\mathrm{id} \odot \theta$ is strong (in $\mathcal{C}$). Let us consider the commutative diagram of figure (**2.2**). The maps $p_U \circ x_P$ and x_S form a cone over $(g \circ \theta, f_U)$, and thus there exists a map $\varphi : X \to Q$ such that $q_U \circ \varphi = p_U \circ x_P$ and $q_S \circ \varphi = x_S$. Then the following diagrams (where the maps are the only maps between those objects that we have considered so far) commute



and thus by unicity of the mediating map, $(\theta \odot \mathrm{id}) \circ \varphi = x_P$. Therefore the square

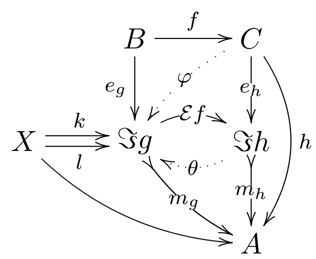


is a pullback. As extremal pas are strong according to lemma (**2.2.7**) and strong maps are preserved by pullback in $\mathcal{C}$, $\theta \odot \mathrm{id}$ is strong too. □

Lemma 2.2.13 :

Let $f : (B, g) \to (C, h)$ in $C \downarrow A$ such that it is a strong map in $\mathcal{C}$. Then $\mathcal{E}f : \Im g \to \Im h$ is an isomorphism.

Δ. Let us consider the following diagram (that we will show to be commutative).



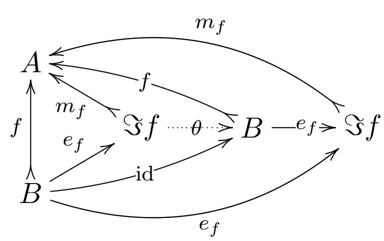
Let us first show that $\mathcal{E}f$ is a monomorphism. Let us suppose that $(\mathcal{E}f) \circ h = (\mathcal{E}f) \circ k$. Thus the lower left part of the diagram commutes and, as m_g is a monomorphism, $k = l$.
As f is a strong map, φ exists and cuts the square in two commutative triangles, thus h factorises through the monomorphism m_g. and therefore, by property of the image, θ exists and makes the two surrounding triangle commute.
But then $m_h \circ (\mathcal{E}f) \circ \theta = m_g \circ \theta = m_h$. As m_h is a monomorphism, θ is right inverse to $\mathcal{E}f$ that is a monomorphism and thus an isomorphism. □

And now the second hypothesis.

Lemma 2.2.14 :

Let $f : B \to A$ be a monomorphism in C, then e_f is an isomorphism.

Δ. Let us consider the following commutative diagram.



As f is a monomorphism, and as it factorises through itself, we have the existence of θ. But then we have $\theta \circ e_f = \mathrm{id}$. Moreover, $e_f \circ \theta$ verifies the same property with respect to the image of f than the identity and thus $e_f \circ \theta = \mathrm{id}$. □

We can now define the bicategory of relations.

Theorem 2.2.15 ($\mathcal{Rel}_{\mathcal{C}}$) **:**

The relations in $\mathcal{C}$ form a bicategory with composition $\Im(\_ \odot \_)$ and identities the identity span.

Δ. It is a consequence of theorem (**2.1.1**). All necessary hypothesis are shown to hold in the preceding lemmas. Indeed, lemma (**2.2.11**) defines the necessary endofunctor on homcats. As e is extremal, lemma (**2.2.12**) implies that $e \odot id$ is strong and lemma (**2.2.13**) that $\mathcal{E}(e \odot id)$ is an isomorphism (and symmetrically with e on the right). Finally, lemma (**2.2.14**) implies that for all $g : C \to A$, $e_{\mathcal{E}(C,g)} = e_{m_g}$ is an isomorphism.
To be sure that all relations are present in the homcats and that the identity is indeed the identity span, we only have to fix the canonical image of a monomorphism to be the monomorphism itself. □

Chapter 3

Polarized spans

The basis of these results were inspired by [FP09], in particular the notion of innocence we use here is theirs.

3.1 Unsynchronised spans

In all that follows, we will work in $\mathcal{E}_p$ (but the same story can be told in $\mathcal{E}_{pr}$).
What we are aiming at is first to define a category of spans where the two maps have disjoint domains. Such spans would not only represent a map into the product, but also a map into the co-product, which is the object they study in [FP09]. But the usual composition of spans do not preserve the disjoint domain property, and so we have to define a "pullback" a little smaller.

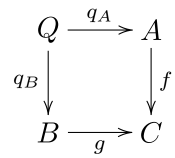
Definition 3.1.1 (Pullback without undefined synchronisation) **:**
Let $f : A \rightharpoonup C$ and $g : B \rightharpoonup C$ be two spans with a same co-domain and let

$$\begin{array}{ccc} P & \xrightarrow{p_A} & A \\ {\scriptstyle p_B}\downarrow & & \downarrow{\scriptstyle f} \\ B & \xrightarrow[g]{} & C \end{array}$$

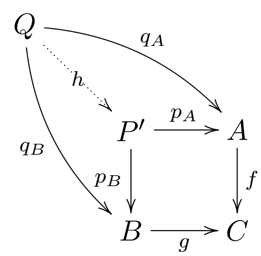
be their pullback. Let P' be the sub event structure of P that consists of all events e such that for all $e' \in \lceil e \rceil$, if $p_A(e')\downarrow$ and $p_B(e')\downarrow$ then $f \circ p_A(e')\downarrow$ (or equivalently $g \circ p_B(e')\downarrow$). As P' is down-closed, p_A and p_B restrict to partial maps of event structures on P'.
P' is called the pullback of f and g without undefined synchronisation.

Theorem 3.1.2 :

The pullback without undefined synchronisation verifies the following "universal" property.
Let us consider the commutative square



where for all $e \in Q$, if $q_A(e)\downarrow$ and $q_B(e)\downarrow$ then $f \circ q_A(e)\downarrow$ (or equivalently $g \circ q_B(e)\downarrow$).
Then there exists a unique map h such that



commutes.

Δ. By universal property of the pullback there exists a unique map $h : Q \rightharpoonup P$ such that the diagram commutes. It suffices to show that $\Im h \subseteq P'$ to have what we want.
Let $e' \leqslant e \in \Im h$. Then there exists $q \in Q$ such that $h(q) = e$. But as $h(\lceil q \rceil)$ is down-closed there exists $q' \in Q$ such that $h(q') = e'$. If $p_A(e')\downarrow$ and $p_B(e')\downarrow$ then as the diagram commutes, $q_A(q')\downarrow$ and $q_B(q')\downarrow$ and thus $f \circ p_A(e') = f \circ q_A(q')\downarrow$. □

Let us now prove two technical lemmas that will be useful later on.

Lemma 3.1.3 :

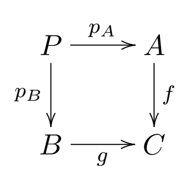
Let P be the pullback without undefined synchronisation of $f : A \rightharpoonup C$ and $g : B \rightharpoonup C$ and let $x \in P$. Let $(a, \_)$ and $(\_, b)$ be maximal elements of these forms in x (we call a a maximal left element of x and b a maximal right). Then it is impossible that $f(a)\uparrow$ and $g(b)\uparrow$.

Δ. First of all, one of $(a, \_)$ and $(\_, b)$ must be the top element of x. If both are, then (a, b) is the top element of x and thus, as P is without undefined synchronisation, $f(a)\downarrow$ and $g(b)\downarrow$.
Let us now suppose that the top element of x is $(a, \star)$. Then $(b, \_) \leqslant (a, \star)$ and thus there exists c such that $(\_, b) \prec (c, \star) \leqslant (a, star)$ as $(\_, b)$ is maximal of this form. But then lemma **(1.4.5)**, implies that $p_A(\_, b)\downarrow$, and thus as P is without undefined synchronisation, $g(b)\downarrow$.
The proof is symmetric if the top element of x is $(\star, b)$. □

Lemma 3.1.4 :

The pullback without undefined synchronisation of a rigid map is rigid.

Δ. Let us consider the following undefined pullback where g is rigid.



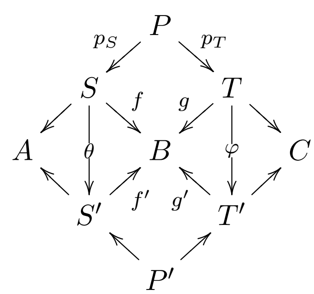
and let x, $y \in P$ such that $x \leqslant y$, $p_A(x)\downarrow$ and $p_A(y)\downarrow$. Let us prove by induction on the number of events between x and y that $p_A(x) \leqslant p_A(y)$.
If $x = y$ the result is trivial.
If $x \prec y$, then lemma (**1.4.5**) implies that $p_A(x) \prec p_A(y)$ (and we have finished) or $p_B(x) \prec p_B(y)$. But then as g is rigid and the diagram commutative, $f \circ p_A(x) \leqslant f \circ p_A(y)$ and thus because of lemma (**1.1.4**), $p_A(x) \leqslant p_A(y)$.
If there exists $z \in P$ such that $x < z < y$ such that $p_A(z)\downarrow$ then by induction $p_A(x) \leqslant p_A(z) \leqslant p_A(y)$.
Finally, if we are in none of the above cases, let x' and $y' \in P$ such that $x \prec x' \leqslant y' \prec y$. Then lemma (**1.4.5**) implies that $p_B(x) \prec p_B(x')$ and $p_B(y') \prec p_B(y)$.
Let us show by induction that $p_B(x') \leqslant p_B(y')$. If they are equal, then it is clear. Else, let $x'' \in P$ such that $x' \prec x''$, then as only $p_B(x'')$ can be defined, lemma (**1.4.5**) implies that $p_B(x') \prec p_B(x'')$ and, as by induction $p_B(x'') \leqslant p_B(y')$ we have what we want.
This implies that $p_B(x) \leqslant p_B(y)$. As P is without undefined synchronisation, we must have $g \circ p_B(x)\downarrow$ and $g \circ p_B(y)\downarrow$. As g is rigid, $g \circ p_B(x) \leqslant g \circ p_B(y)$ and thus by commutativity, $f \circ p_A(x) \leqslant f \circ p_A(y)$. Therefore, as lemma (**1.1.4**) states that f must reflect order on configurations, $p_A(x) \leqslant p_A(y)$. □

Now that we have defined a modified "pullback" with the right properties, we can define the composition that interests us, the one that will preserve the disjoint domain property. And of course show that it defines a new bicategory of spans.

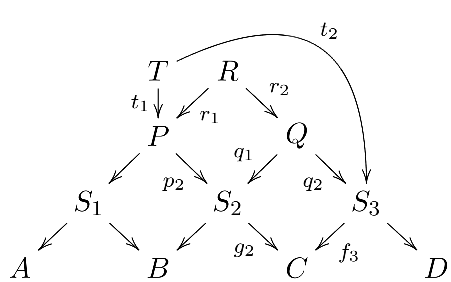
Definition 3.1.5 (Unsynchronised composition) **:**
Let $s_1 : A \leftrightarrows B$ and $s_2 : B \leftrightarrows C$ be two spans, their unsynchronised composition $s_2 \oplus s_1 : A \leftrightarrows C$ is built according to figure (2.1) where P is the pullback without undefined synchronisation.

Definition 3.1.6 ($\mathcal{S}pan_{\oplus}$) **:**
Let $\mathcal{S}pan_{\oplus}$ be the bicategory where 0-cells are event structures, 1-cells are spans, 2-cells are span morphism. The span compose according to $\oplus$ and the identity are the identity spans.

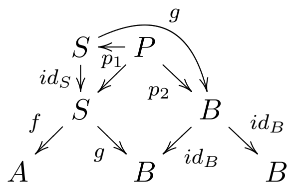
Δ. First let us consider $\theta : s \Rightarrow s'$ and $\varphi : t \Rightarrow t'$ and thus the following diagram



Let us suppose we have $x \in P$ such that $\theta \circ p_S(x)\downarrow$ and $\varphi \circ p_T(x)\downarrow$. Then as P is without undefined synchronisation $f \circ p_S(x)\downarrow$ and $g \circ p_T(x)\downarrow$. As the diagram commutes $f' \circ \theta \circ p_S(x)\downarrow$ and $g' \circ \varphi \circ p_T(x)\downarrow$ and thus there is a span morphism from $t \oplus s$ to $t' \oplus s'$, let us call it $\varphi \oplus \theta$. The proof that $\oplus$ is a functor is the same that when one considers a normal pullback.
Let us now consider s_1, s_2 and s_3 that are composable and let us consider the diagram



Let $x \in R$ such that $r_1(x)\downarrow$ and $q_2 \circ r_2(x)\downarrow$. Then, as R is without undefined synchronisation, $q_1 \circ r_2(x)\downarrow$ and thus $f_3 \circ q_2 \circ r_2(x)\downarrow$. Thus there is a unique map from R to T such that the right triangles commutes (and thus it is a span morphism).
Let now $x \in T$ such that $p_2 \circ t_1(x)\downarrow$ and $t_2(x)\downarrow$. Then, as $T$is without undefined synchronisation, $f_3 \circ t_2(x)\downarrow$. Thus there is a unique map h from T to Q such that the right triangles commutes. In particular $q_1 \circ h = p_2 \circ t_1$.
Let $x \in T$ such that $h(x)\downarrow$ and $t_1(x)\downarrow$. Because of lemma (**1.4.4**), $q_1 \circ h(x)\downarrow$ or $q_2 \circ h(x)\downarrow$. Let us suppose $q_2 \circ h(x)\downarrow$, then as $q_2 \circ h = t_2$ and T is without undefined synchronisation, $g_2 \circ p_2 \circ t_1(x)\downarrow$ and thus for the diagram to commute $g_1 \circ h(x)\downarrow$. Thus in all cases $g_1 \circ h(x)\downarrow$ and thus there is a unique map from T to R such that the right triangles commute (and thus it is a span morphism).
The proof that those two span morphism are inverse and that the isomorphism is natural is the same as when one considers a normal pullback. As the situation is symmetric we deduce a natural isomorphism between $s_3 \obar (s_2 \obar s_1)$ and $(s_3 \obar s_2) \obar s_1$. The fact that the pentagon commutes is also given by the usual proof.
Let us now consider



Let $x \in S$ such that $g(x)\downarrow$, then $g \circ id_S(x)\downarrow$ and thus there is a unique map from S to P such that the right triangles commutes. The proof that its inverse is p_1 and that they form a natural isomorphism is the same as when P is the usual pullback. The existence of the isomorphism with the identity on the left is symmetrical. Finally the commutativity of the triangle is proved as if we were using usual pullbacks. □

Disjoint domain spans now make sense, as will show the following lemma.

Definition 3.1.7 (Disjoint domain spans) **:**
Let $s := (f, S, g)$ be a span in $\mathcal{E}_p$. It is said to be a disjoint domain span if $\mathfrak{D}_f \cap \mathfrak{D}_g = \emptyset$.

Lemma 3.1.8 :

> Let s_1 and s_2 be two composable spans such that s_1 (or s_2) is a disjoint domain span, then $s_2 \obar s_1$ is a disjoint domain span.

Δ. Let $x \in \mathfrak{D}_{f_1 \circ p_S} \cap \mathfrak{D}_{f_2 \circ p_T}$, then it implies a fortiori that $p_S(x)\downarrow$ and $p_T(x)\downarrow$ and thus that $g_1 \circ p_S(x)\downarrow$. But that implies that $p_S(x) \in \mathfrak{D}_{f_1} \cap \mathfrak{D}_{g_1}$ and thus that s_1 is not a disjoint domain span. Moreover the symmetric proof shows that s_2 is not either. □

We will now define, and prove the composition of, the notion of determinacy in a span. It states that in a deterministic span, no consistent output can come from an inconsistent input.

Definition 3.1.9 (Deterministic span) **:**
A span $s := (f, S, g)$ is deterministic, if for all $X \subseteq_f S$ down-closed, such that $f(X)$ is a configuration, then X is a configuration.

Theorem 3.1.10 :

> Let s_1 and s_2 be two composable deterministic spans, then $s_2 \obar s_1$ is a deterministic span.

Δ. Let us consider the diagram of figure (**2.1**).
Let $X \subseteq_f P$ down-closed, such that $f_1 \circ p_S(X) \in \mathcal{C}^o(A)$. Let us first prove that $p_S(X)$ is down-closed. Let $p_S(x) \in p_S(X)$ and $y \leqslant_S x$ then as $p_S(\lceil x \rceil)$ is a configuration it contains $\lceil p_S(x) \rceil$ and thus $y \in p_S(\lceil x \rceil)$, but that implies that there exists $z \in \lceil x \rceil \subseteq X$ such that $y = p_S(z)$. Thus $y \in p_S(X)$. Symmetrically, $p_T(X)$ is down-closed.

Moreover, $f_1(p_S(X)) \in \mathcal{C}^o(A)$ and thus, as s_1 is deterministic, $p_S(X) \in \mathcal{C}^o(S)$. But that implies that $f_2(p_T(X)) = g_1(p_S(X)) \in \mathcal{C}^o(B)$ and thus, as s_2 is deterministic, $p_T(X) \in \mathcal{C}^o(T)$.
To show that X is consistent (and thus a configuration), we have to show that $\bigcup X \in \mathcal{C}^o(S) \times \mathcal{C}^o(T)$. As P is a sub event structure of the pullback, that is itself a sub event structure of the product, that will then imply that $X \in \mathcal{C}^o(P)$.
First $\pi_S(\bigcup X) = \bigcup_{x\in X} \pi_S(x)$. Let us show that it is equal to $p_S(X)$. Let $x \in X$, then it is of the form $\lceil e \rceil_y$ for some e and y. Then $p_S(x) \doteq \eta^{-1}(\lceil \pi_S(e) \rceil_{\pi_S(y)}) \doteq \pi_S(e) \in \pi_S(\lceil e \rceil_y) = \pi_S(x)$. Thus $p_S(X) \subseteq \bigcup_{x\in X} \pi_S(x)$. Let now $x \in X$ and $e \in x$, then $\lceil e \rceil_x \subseteq x$ and as X is down-closed, $\lceil e \rceil_x \in X$. Moreover, $\pi_S(e) \doteq p_S(\lceil e \rceil_x)$ and thus $\bigcup_{x\in X} \pi_S(x) \subseteq p_S(X)$.
As we have show earlier, $\pi_S(X)\mathcal{C}^o(S)$ and thus $\pi_S(\bigcup X) \in \mathcal{C}^o(S)$. Symmetrically, $\pi_T(\bigcup X) \in \mathcal{C}^o(T)$.
Let us now consider $e_1, e_2 \in \bigcup X$. Then there exists $x_1, x_2 \in X$ such that $e_1 \in x_1$ and $e_2 \in X_2$. If $e_2 \notin x_1$ then x_1 is such $e_1 \in x_1 \iff e_2 \notin x_1$. Moreover, as x_1 is a configuration of the product, by definition, $\pi_S(x_1) \in \mathcal{C}^o(S)$ and $\pi_T(x_1) \in \mathcal{C}^o(T)$. If e_2 and $e_1 \in x_1$, then by definition of x_1 there exists $y \subseteq x_1 \subseteq \bigcup X$ such that $\pi_S(y) \in \mathcal{C}^o(S)$, $\pi_T(y) \in \mathcal{C}^o(T)$ and $e_1 \in y \iff e_2 \notin y$.
Finally, let us suppose $\pi_S(e_1) = \pi_S(e_2)$ (and both defined), then if e_1 is of the form (s_1, t_1) because $X \subseteq P$, it implies that $g_1(s_1)\downarrow$. But then $g_1(p_S(e_2))\downarrow$ too and thus $f_2(p_T(e_2))\downarrow$. In particular, $p_T(e_2)\downarrow$. But then we have $f_2(p_T(e_2)) = g_1(p_S(e_2)) = g_1(s_1) = f_2(t_1)$ and thus, as f_2 is locally injective, $p_T(e_2) = t_2$. If e_1 is of the form $(s_1, \star)$ then, because the previous proof is symmetric, e_2 must be of the form $(s_2, \star)$. And thus in both cases $e_1 = e_2$.
As S and T play symmetric roles, p_T is injective too and $\bigcup X$ is a configuration of the product. □

3.2 Polarisation

As we have stated in our introduction, the games we consider are between two opponents, that have their own moves in the game. To mark to whom belongs each move, will use a polarisation function. Positive moves will be program (player) moves and negative will be environment (opponent) moves. There is also a neutral polarisation to mark internal calculations (in particular those that appear during composition).

Definition 3.2.1 (Polarised event structure) **:**
A polarised event structure (E, θ_E) is an event structure E along with a total map $\theta_E : E \to \{-, \diamond, +\}$. It is said to be strictly polarised when for all $e \in E$, $\theta_E(e) \neq \diamond$.
A partial map $f : (E, \theta_E) \rightharpoonup (F, \theta_F)$ of polarized event structure is a partial map of event structures such that for all $e \in E$ such that $f(e)\downarrow$ and $\theta_E(e) \neq \diamond$, $\theta_F(f(e)) = \theta_E(e)$.
It is said to be strict if the equality also holds if $\theta_E(e) = \diamond$.

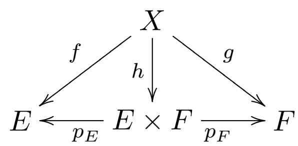
Polarised event structures and partial maps form a category $\mathcal{E}_p^{\pm}$. Let us describe the product in this category.

Definition 3.2.2 (Product in $\mathcal{E}_p^{\pm}$) **:**
Let (E, θ_E) and (F, θ_F) be two polarised event structures, let $x \in E \times F$ and let $\theta_{E\times F}(x)$ be given according to $\theta_E(p_E(x))$ and $\theta_F(p_F(x))$ and the following table

	$+$	$-$	$\diamond$	$\uparrow$
$+$	$+$	$\diamond$	$\diamond$	$+$
$-$	$\diamond$	$-$	$\diamond$	$-$
$\diamond$	$\diamond$	$\diamond$	$\diamond$	$\diamond$
$\uparrow$	$+$	$-$	$\diamond$	

Theorem 3.2.3 :
The polarised event structure $(E \times F, \theta_{E\times F})$ is the product of E and F in $\mathcal{E}_p^{\pm}$.

Δ. First, the table has been built such that p_S and p_T are maps of polarised event structures.
Second let us consider two maps $f : X \rightharpoonup E$ and $g : X \rightharpoonup F$ in $\mathcal{E}_p^{\pm}$. Then, as $E \times F$ is a product in $\mathcal{E}_p$, we have a unique mediating map h such that



commutes. We only need to show that this map h respects polarisation.
Let $x \in X$ such that $\theta_X(x) = +$, then if $f(x)\downarrow$ and $g(x)\downarrow$ then $\theta_E(f(x)) = +$ and $\theta_F(g(x)) = +$ and thus, because the diagram commutes $\theta_E(p_E(h(x))) = \theta_F(p_F(h(x))) = +$ and thus $\theta_{E\times F}(h(x)) = +$. Now, if one of them is undefined, let us say $g(x)\uparrow$ and $f(x)\downarrow$, then because of commutativity $p_F(h(x))\uparrow$ and $\theta_E(p_E(h(x))) = +$ and thus $\theta_{E\times F}(h(x)) = +$. Finally, if both are undefined, then $p_E(h(x))\uparrow$ and $p_F(h(x))\uparrow$ but as, according to lemma (**1.4.4**), $E \times F$ does not contain elements where both projections are undefined, $h(x)\uparrow$.
The same holds if $\theta_X(x) = -$. □

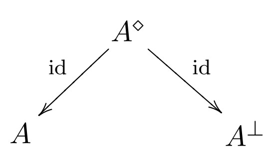
This product allows us to define pullbacks without undefined synchronisation in $\mathcal{E}_p^{\pm}$ as previously, and thus $\mathcal{S}pan_{\oplus}^{\pm}$, the category of unsynchronised spans in $\mathcal{E}_p^{\pm}$.
As one could guess, adding polarity leads us naturally to defining a dual. It will allow us in particular to define maps that reverse parity.

Definition 3.2.4 ($\_^{\perp}$) **:**
Let (A, θ_A) be a polarized event structure. Let $+^{\perp} := -$, $-^{\perp} := +$ and $\diamond^{\perp} := \diamond$, finally let $A^{\perp}$ be the polarised event structure $(A, \theta_A^{\perp})$.
Moreover, let $f : A \rightharpoonup B$ be a partial map of polarised event structure, then f is also a map of polarised event structure from $A^{\perp}$ to $B^{\perp}$, noted $f^{\perp}$.
It is evident that $\_^{\perp}$ is a functor. Furthermore, it is an involution.

Let us now define polarised spans. They are our first step toward strategies.

Definition 3.2.5 (Polarised spans) **:**
Let A and B be two strictly polarised event structures. A polarized span from A to B is a disjoint domain span $(f, S, g) : A^{\perp} \leftrightarrows B$ such that f and g are strict and such that if $x \notin \mathfrak{D}_f \cup \mathfrak{D}_g$ then $\theta_S(x) = \diamond$.

Definition 3.2.6 (Composition of polarised spans) **:**
Let (A, θ_A) be a polarised event structure, and let $A^{\diamond}$ be $(A, \theta_{\diamond})$ where for all $a \in A$, $\theta_{\diamond}(a) = \diamond$. Then the identity is a polarised map from $A^{\diamond}$ to A (or to $A^{\perp}$). Let us write $s_A : A \leftrightarrows A^{\perp}$ for the span



Let now $s_1 : A^{\perp} \leftrightarrows B$ and $s_2 : B^{\perp} \leftrightarrows C$ be two spans in $\mathcal{E}_p^{\pm}$. Then $s_2 \boxbar s_1 := s_2 \oplus s_B \oplus s_1$ (one may choose which ever way to put the parenthesis, it is the same up to span isomorphism).

One may notice that this composition can also be seen as

1. Compose the two spans in $\mathcal{E}_p^{\pm}$ forgetting about polarity (thus the left event structure of s_1 and the right of s_2 are indeed the same)

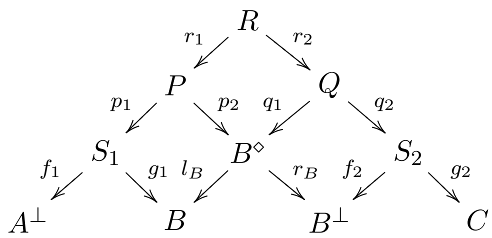


Figure 3.1: Composition of polarised spans

2. Put the polarity back on the pullback without undefined synchronisation using the table given before

We need, of course, to show that polarised spans are stable under the composition we just defined.

Lemma 3.2.7 :

Let s_1 and s_2 be two composable polarised spans. Then $s_1 \boxbar s_2$ is polarised.

Δ. By lemma (**3.1.8**), $s_1 \boxbar s_2$ is a disjoint domain span. We only have to show that the two arrows of the span are strict. Let us consider the diagram of figure (**3.1**).
Let $x \in R$, if $r_1(x)\downarrow$ and $r_2(x)\downarrow$, then $p_2 \circ r_1(x)\downarrow$ and $q_1 \circ r_2(x)\downarrow$ and are equal. And thus as in $B^\diamond$ everything has polarity $\diamond$, $\theta_P(r_1(x)) = \diamond$ and $\theta_Q(r_2(x)) = \diamond$. Thus $\theta_R(x) = \diamond$. Moreover, as l_B is total and the squares commute, $g_1 \circ p_1 \circ r_1(x)\downarrow$ and thus as s_1 is a disjoint domain span $f_1 \circ p_1 \circ r_1(x)\uparrow$. For symmetric reason $g_2 \circ q_2 \circ r_2(x)\uparrow$.
Now if $r_2(x)\uparrow$, then because of lemma (**1.4.4**), we have $r_1(x)\downarrow$. If $p_2 \circ r_1(x)\downarrow$, then for the square to commutes, we must have $r_2(x)\downarrow$ which contradicts the hypothesis, thus $p_2 \circ r_1(x)\uparrow$. But then, once more because of lemma (**1.4.4**), $p_1 \circ r_1(x)\downarrow$. Now, if $\theta_S(p_1 \circ r_1(x)) = \diamond$, then $\theta_R(x) = \diamond$, and, as s_1 is a polarised span, $f_1 \circ p_1 \circ r_1(x)\uparrow$, or else it would have polarity $\diamond$, but there is no $\diamond$ in A.
If $\theta_S(p_1 \circ r_1(x)) = +$ then $\theta_R(x) = +$. If $g_1 \circ p_1 \circ r_1(x)\downarrow$, then for the square to commute, we would have $p_2 \circ r_1(x)\downarrow$, but we have already shown this impossible. Thus $g_1 \circ p_1 \circ r_1(x)\uparrow$, but because s_1 is polarised, we must have $f_1 \circ p_1 \circ r_1(x)\downarrow$. As the case is symmetric when $\theta_S(p_1 \circ r_1(x)) = -$, and symmetric if $r_1(x)\uparrow$, we have proved that $s_2 \boxbar s_1$ is polarised. □

Let us now define a number of properties that can have the polarised spans, and show that each composes. The first of them, innocence is the most important, because it will be central in defining the category $\mathcal{S}pan_{Inn}$. The other will only be needed to characterise the spans that represent simple games in chapter 5.

Definition 3.2.8 (Innocence) **:**
A polarised span $s := (f, S, g)$ is innocent if for all x, $y \in S$ such that $x \prec y$ and $\theta(x) = +$ or $\theta(y) = -$, then there exists h among f and g such that $h(x)\downarrow$, $h(y)\downarrow$ and $h(x) \prec h(y)$.

Lemma 3.2.9 :

Let s_1 and s_2 be two composable innocent spans, then $s_2 \boxbar s_1$ is an innocent span.

Δ. Let us consider the diagram of figure (**3.1**) and let x, $y \in R$ such that $x \prec y$ and $\theta_R(y) = -$, the proof of lemma (**3.2.7**) has shown that $f_1 \circ p_1 \circ r_1(y)\downarrow$ and that $r_2(y)\uparrow$, $p_2 \circ r_1(y)\uparrow$, and $g_1 \circ p_1 \circ r_1(y)\uparrow$ (or symmetrically on the right).
Lemma (**1.4.5**) applied to R and P implies that $p_1 \circ r_1(x) \prec p_1 \circ r_1(y)$. But as $\theta_S(p_1 \circ r_1(y)) = -$, innocence of s_1 implies that $f_1 \circ p_1 \circ r_1(x) \prec f_1 \circ p_1 \circ r_1(y)$.
The proof is identical (reversing the role of x and y) if $\theta_R(x) = +$. □

Definition 3.2.10 (Filiform event structures) **:**
An event structure E is said to be filiform if for all $e \in E$, $\lceil e \rceil$ is totally ordered.

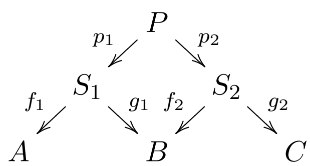


Figure 3.2: The othe way to see the composition

A span (f, S, g) is said to be filiform if S is.

One may notice that an event structure is filiform, if and only if each event has at the most one direct predecessor. The full subcategory of filiform event structure in $\mathcal{E}_p$ is called $\mathcal{F}il_p$ and that of $\mathcal{E}_{pr}$ is called $\mathcal{F}il_{pr}$.

Lemma 3.2.11 :

let s_1 and s_2 be two innocent and filiform spans. Then $s_1 \boxbar s_2$ is filiform.

Δ. Let us consider the diagram of figure (**3.2**) and let x, y and z in P such that x and y are both direct predecessors of z. Then lemma (**1.4.5**) implies the projection of x must be directly before that of z on one side (and similarly for y).
Let us suppose first that $p_1(x) \prec p_1(z)$ and $p_1(y) \prec p_1(z)$ (or symmetrically on the right). Then as s is filiform, we must have $p_1(x) = p_1(y)$ and as x and y are consistent, local injectivity implies that $x = y$.
Now if $p_1(x) \prec p_1(z)$ but $p_2(y) \prec p_2(z)$, then $g_1 \circ p_1(z)$ and $f_2 \circ p_2(z)$ are both defined and equal, and as f_2 reverses parity (it goes into the dual) and B is strictly polarised, one of $p_1(z)$ and $p_2(z)$ is negative. Let us suppose $p_2(z)$ is. Then, as s_2 is innocent $f_2 \circ p_2(x)$ is defined and thus, for the square to commute, $g_1 \circ p_1(x) \downarrow$ and $g_1 \circ p_1(x) \prec g_1 \circ p_1(z)$. But then lemma (**1.1.4**) implies that $p_1(x) \leqslant p_1(z)$ and lemma (**1.1.5**) implies that $p_1(x) \prec p_1(z)$. Thus we are back in the first case. □

Definition 3.2.12 (Alternation) **:**
A polarised event structure E is said to be alternating if for all non neutral x, $y \in E$ such that all events between them are neutral, then $\theta_E(x) = \theta_E(y)^\perp$.
A polarised span (f, S, g) is said to be alternating if S is.

Lemma 3.2.13 :

Let A and B be two alternating strictly polarised event structure and let (f, S, g) be an innocent span from A to B. Then S is alternating.

Δ. Let us suppose we have x and $y \in S$ be both positive such that $x \leqslant y$ and there are only neutral between them. Then let $z \in E$ such that $x \prec z \leqslant y$. Then as the span is innocent, we can suppose $g(x) \prec g(z)$ (the other case is identical). As $g(x)$ is positive and B is strict and alternating, $h(z)$ is negative. But then z must be negative too. This contradicts the fact that z is neutral or positive.
The case when x and y are both negative is treated in the same manner. □

Definition 3.2.14 (Concreteness) **:**
A polarised event structure is said to be concrete if all pairs of positive events with a common direct predecessor are in conflict.
A polarised span (f, S, g) is said to be concrete if S is.

Lemma 3.2.15 :

Let s_1 and s_2 be two innocent and concrete spans, then $s_2 \boxbar s_1$ is concrete.

Δ. Let us consider the diagram of figure (**3.1**). Let x, $y \in R$ both positive, such that they have a common direct predecessor z. The proof of lemma (**3.2.7**) has shown that $r_1(x)$ and $r_2(x)$ cannot be both defined (and identically for y).
Let us first suppose that $r_1(x)\downarrow$ and $r_1(y)\downarrow$. Then lemma (**1.4.5**) implies that $p_1 \circ r_1(z) \prec p_1 \circ r_1(x)$ and $p_1 \circ r_1(z) \prec p_1 \circ r_1(y)$. Moreover $p_1 \circ r_1(x)$ and $p_1 \circ r_1(y)$ are both positive and thus, as s_1 is concrete, $p_1 \circ r_1(x)$ and $p_1 \circ r_1(y)$ are in conflict. Therefore x and y are in conflict.
The case on the right is identical, let us now consider the case when $r_1(x)\downarrow$ and $r_2(y)\downarrow$ (or symmetrically). Lemma (**1.4.5**) implies that $p_1 \circ r_1(z) \prec p_1 \circ r_1(x)$ and $q_2 \circ r_2(z) \prec q_2 \circ r_2(y)$. But then $p_2 \circ r_1(x)\downarrow$ and thus $l_B \circ p_2 \circ r_1(x)$ and $r_B \circ p_2 \circ r_1(x)$ must have different polarities, thus one of them must be positive. We can suppose without loss of generality that $l_B \circ p_2 \circ r_1(x)$ is, then as $g_1 \circ p_1 \circ r_1(z) = l_B \circ p_2 \circ r_1(z)$, $p_1 \circ r_1(z)$ is positive too. But as s_1 is innocent, we must have $g_1 \circ p_1 \circ r_1(x)\downarrow$. But the proof of lemma (**3.2.7**) shows that, as x is positive and $r_1(x)\downarrow$, then $f_1 \circ p_1 \circ r_1(x)\downarrow$, this contradicts the fact that s_1 is a disjoint domain span. □

Definition 3.2.16 (Rigid span) **:**
A polarised span (f, S, g) is said to be rigid if f and g are.

Lemma 3.2.17 :

Let s_1 and s_2 be two rigid spans, then $s_2 \boxminus s_1$ is rigid.

Δ. This result is an immediate consequence of lemma (**3.1.4**), i.e. that rigid map are preserved by pullback without undefined synchronisation, and the fact that rigid maps compose. □

3.3 The identity problem, or why restrict to visible parts only

Disjoint domain spans cannot be seen as a sub bicategory of $\mathcal{S}pan_{\oplus}$, indeed the identity span has not disjoint domains. Moreover, there is no identity for disjoint domain spans in general, we have to restrict to innocent ones. But then remains the problem that $\oplus$ produces internal synchronisation events that we have to get rid of to have identities, thus we need to change a little the composition we use.

Definition 3.3.1 (Visibility and the $\mathcal{V}$ functor) **:**
Let $s := (f, S, g)$ be a span in $\mathcal{E}_p$. We say that an event $e \in S$ is visible if either $f(e)\downarrow$ or $g(e)\downarrow$. The span s is said to be fully visible if all the elements of S are visible.
Let S' be the sub event structure of S that contains all the visible events and let us write $\mathcal{V}s := (f_{|S'}, S', g_{|S'})$ for the visible part of s. It is indeed a span in $\mathcal{E}_p$, it is called the visible part of s.
Moreover, let $\theta : s \Rightarrow t$ and let $e \in S$ be visible. Then $\theta(e)$ must be defined for the diagram to commute and it must be visible. Thus θ restricts to a span morphism $\mathcal{V}\theta : \mathcal{V}s \Rightarrow \mathcal{V}t$.
Finally, the map from S to S' that is the identity on visible objects is a span morphism, that we call $\wp$, from s to s' that is obviously natural in s.

One may remark that we have proved that for all span morphism θ, $\mathcal{V}\theta$ is total. This endofunctor $\mathcal{V}$ will be used in a extremity construction, thus we have to prove the two hypothesis.

Lemma 3.3.2 :

Let s be a span, then $\wp_{\mathcal{V}s}$ is an isomorphism.

Δ. Indeed, $\mathcal{V}s$ as only visible events and thus restricting to visible events does not change the span, thus $\wp_{\mathcal{V}s}$ is the identity. □

The second one needs a little more work. The following lemmas are technical results that help show it. We shall admit them, as they need detailed, and rather uninteresting, proofs.

Lemma 3.3.3 :
Let $e \in \mathcal{E}_p^{\pm}(A, B)$. It is an extremal map if and only if it is rigid, strictly polarised and surjective on finite configurations.

Lemma 3.3.4 :
Let $e \in \mathcal{E}_p^{\pm}(A, B)$. It is a monomorphism if and only if it is injective on configurations.

Lemma 3.3.5 :
Extremal maps are preserved under pullback without undefined synchronisation.

Lemma 3.3.6 :
Let θ be span morphism that is extremal, then $\mathcal{V}\theta$ is extremal too.

Lemma 3.3.7 :
The pullback without undefined synchronisation of an injection is an injection.

Lemma 3.3.8 :
Let θ be span morphism that is injective, then $\mathcal{V}\theta$ is injective too, and thus a monomorphism.

We can now show the second hypothesis of the extremity construction.

Lemma 3.3.9 :
Let s and t be two spans, then $\mathcal{V}(\wp_s \obar \mathrm{id}_t)$ and $\mathcal{V}(\mathrm{id}_t \obar \wp_s)$ are isomorphisms.

∆. Let us consider the diagram of figure (**2.2**) where $\theta = \wp$. Let us suppose that for all $y \in X$, if $x_S(y)\downarrow$ and $x_P(y)\downarrow$ then $p_T \circ x_P(y)\downarrow$. Then let us suppose that we have $y \in X$ such that $x_S(y)\downarrow$ and $p_U \circ x_P(y)\downarrow$, then $p_T \circ x_P(y)\downarrow$ and thus, as P is without undefined synchronisation, $f_U \circ p_U \circ x_P(y)\downarrow$. Thus there exists a map $\varphi : X \to Q$ such that $q_U \circ \varphi = p_U \circ x_P$ and $q_S \circ \varphi = x_S$. The rest of the proof proceeds like in lemma (**2.2.12**) to show that this square is a pullback

$$\begin{array}{ccc} Q & \xrightarrow{\wp \obar \mathrm{id}} & P \\ {\scriptstyle q_S}\downarrow & & \downarrow{\scriptstyle p_T} \\ S & \xrightarrow[\wp]{} & T \end{array}$$

As it is clear that $\wp$ is injective and an extremal map (see characterisation of lemma (**3.3.3**)) and thus, according to the list of preservation lemmas (from (**3.3.5**) to (**3.3.8**)), $\mathcal{V}(\wp \obar \mathrm{id})$ is a monomorphism and an extremal map. It is therefore an isomorphism. □

We can now define the composition (and the bicategory that goes with it) that correspond in game semantics literature to parallel composition (here, taking the pullback without undefined synchronisation) and hiding (here, forgetting the invisible events).

Definition 3.3.10 ($\mathcal{VS}pan_{\obar}^{\pm}$) **:**
Let $\mathcal{VS}pan_{\obar}^{\pm}$ be the bicategory where 0-cells are polarised event structures, 1-cells are fully visible spans and 2-cells are span morphisms. The composition is $s \circledast t := \mathcal{V}(s \obar t)$ and the identities are the identity spans.

∆. The existence of this bicategory follows from theorem (**2.1.1**). The necessary hypothesis are proven in lemmas (**3.3.9**) and (**3.3.2**). □

In this category of fully visible spans, we have a composition of polarised spans, that we write $\boxast$.
Even with hiding, innocence will not be enough to have an identity for polarised spans and the $\boxast$ composition. We need to introduce two new span properties, and as usual show that they compose.

Definition 3.3.11 (Negative saturation) **:**
Let $s := (f, S, g)$ be a span of polarised event structures from A to B, we say that s is negatively saturated when, for all $X \in \mathcal{C}^o(S)$, if $f(X)$ has a negative extension, i.e. there exists $Y \in \mathcal{C}^o(A^\perp)$ such that $X \subseteq Y$ and $\theta_{A^\perp}(Y \setminus f(X)) = \{-\}$, then there exists $X' \in \mathcal{C}^o(S)$ such that $X \subseteq X'$ and $f(X') = Y$; and identically for g.

Lemma 3.3.12 :

Let s_1 and s_2 be two composable innocent negatively saturated spans, then $s_2 \boxminus s_1$ is negatively saturated.

Δ. Let us consider the composition diagram of figure (3.2). Let $X \in \mathcal{C}^o(P)$ such that $f_1 \circ p_1(X)$ has a negative extension Z in $A^\perp$ (the case on the right is symmetric, if not a little simpler because their is no dual to account for). But then $p_1(X) \in \mathcal{C}^o(S_1)$ and thus as s_1 is negatively saturated, there exists $Y \in \mathcal{C}^o(S)$ such that $p_1(X) \subseteq Y$ and $f_1(Y) = Z$. Let us suppose that Y is a minimal such configuration.
Let $y \in Y \setminus p_1(X)$. Then there is no $y' \in y$ such that $y \leqslant y'$ and $f_1(y')\downarrow$, then Y would not be minimal, $Y' = Y \setminus \{y' \in Y \mid y \leqslant y'\}$ would also be a configuration such that $f_1(Y') = Z$. Thus there is a $y' \in Y$ such that $y \leqslant y'$ and $f_1(y')\downarrow$. If $f_1(y)\uparrow$, then there exists $z, z' \in Y$ such that $y \leqslant z \prec z' \leqslant y'$, $f_1(z)\uparrow$ and $f_1(z')\downarrow$. But then as $y \in Y \setminus p_1(X)$, $f_1(z') \in Z \setminus f_1 \circ p_1(X)$ and thus it is negative. as f_1 preserves polarity (into the dual), z' is also negative. Thus the innocence of s_1 is contradicted. Therefore $f_1(y)\downarrow$.
Let $X' := X \cup \{(y, \star) \mid y \in Y \setminus p_1(X)\}$, it is a configuration of the pullback without undefined synchronisation that verifies $X \subseteq X'$ and $f_1 \circ p_1(X') = f_1(Y) = Z$. □

Definition 3.3.13 (Negative coincidence) **:**
Let $s := (f, S, g)$ be a span of polarised event structures and let $x, y \in S$. We say that x and y are coincident if $\lfloor x \rfloor = \lfloor y \rfloor$, $f(x) \doteq f(y)$ and $g(x) \doteq g(y)$.
We therefore say that a span is negative coincidence free if there are no distinct events that are both negative and coincident.

Lemma 3.3.14 :

Let s_1 and s_2 be two composable innocent negative coincidence free spans, then $s_2 \boxminus s_1$ is negative coincidence free.

Δ. Let us consider the composition diagram of figure (3.2). Let $x, y \in P$ negative with the same strict history, such that $f_1 \circ p_1(x)$ and $f_1 \circ p_1(y)$ both defined and equal (once more the case on the right is the same).
Then, let us show that $p_1(x)$ and $p_1(y)$ have the same strict history. Let $z \prec p_1(x)$, then as $p_1(x)$ is negative, innocence implies that $f_1(z)\downarrow$ and that $f_1(z) \prec f_1 \circ p_1(x) = f_1 \circ p_1(y)$. Thus there is a $z' \leqslant p_1(y)$ such that $f_1(z') = f_1(z)$.
But then there are $t, t' \in \lfloor x \rfloor = \lfloor y \rfloor$ such that $p_1(t) = z$ and $p_1(t') = z'$. These two events have the same image by $f_1 \circ p_1$ and are consistent, thus they are equal. Therefore $z = z'$ and $z \leqslant p_1(y)$. We have therefore proved that $\lfloor p_1(x) \rfloor \subseteq \lfloor p_1(y) \rfloor$. As the problem is symmetric, we have indeed that $\lfloor p_1(x) \rfloor = \lfloor p_1(y) \rfloor$. Thus they are negative and coincident, they must be equal.
By hypothesis, we knew that x and y were composed of the same elements except for their top one (let us remember that x and y are prime of the product of stable families, thus sets). We have proved that their top element is $(p_1(x), \star) = (p_1(y), \star)$, and therefore $x = y$. □

We have not shown so far that taking the visible part of a span preserved all the properties that span can have. Let us do this now.

Lemma 3.3.15 :

Let s be a polarised span then its visible part is a polarised span. Moreover, all the following properties are preserved when going from a span to its visible part : innocent, negatively saturated, negative coincidence free, deterministic, filiform, concrete.

Δ. All these properties concern the events on which the span is defined. Forgetting the internal events thus do not change a thing. □

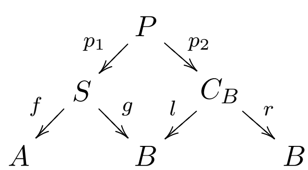
As the identity span is not a polarised span, we have not defined yet what may be the identity for polarised spans. It is the copycat span that we define now.

Definition 3.3.16 (Copycat span) **:**
Let (A, θ_A) be a strictly polarised event structure. Let e in A, we will note $e^\perp$ the element e of $A^\perp$ and symmetrically for $e \in A^\perp$.
Let $C_A := A^\perp \sqcup A$, let $l_A : C_A \rightharpoonup A^\perp$ be the map that is the identity on $A^\perp$ and undefined on A and $r_A : C_A \rightharpoonup A$ be the map that is the identity on A and undefined on $A^\perp$.
Let $\leqslant_{C_A}$ be the transitive closure of $\leqslant_A$, $\leqslant_{A^\perp}$ and $a^- \leqslant_{C_A} a^+$ for all $a \in C_A$. Let $X \subseteq C_A$, then $X = Y \sqcup Z$ where $Y \subseteq A^\perp$ and $Z \subseteq A$. If we forget the polarities, Y and Z are both subsets of A. We say that X is consistent if $Y \cup Z \in \mathfrak{C}_A$. This defines an event structure on C_A.
Then $\mathfrak{cc}_A := (l_A, C_A, r_A)$ is a fully visible polarised span, innocent, negatively saturated and negative coincidence free.

Let us now show what is probably the most technical lemma of this report, but at the same time a fundamental one. It characterises the spans that are interesting, that is those for which copycat is the identity.

Lemma 3.3.17 :
Let s be a fully visible polarised span. Then $vis(s \circledast \mathfrak{cc})$ and $\mathcal{V}(\mathfrak{cc} \circledast s)$ are isomorphic to s if and only if s is innocent, negatively saturated and negative coincidence free.

Δ. Let us consider the following composition diagram (we consider the composition as in the remark following definition **(3.2.6)**, for the sake of simplicity).



The proof of lemma **(3.2.7)** has shown that the elements in P are of four types : $\lceil (e, b) \rceil_X$, $\lceil (e, \star) \rceil_X$, $\lceil (\star, b^+) \rceil_X$ and $\lceil (\star, b^-) \rceil_X$. We will define a map $\theta : P \to S$ considering those four cases.
First, $\theta(\lceil (e, b) \rceil_X) \uparrow$. Secondly, $\theta(\lceil (e, \star) \rceil_X) = e$.
Thirdly, let us consider an element of the form $\lceil (\star, b^+) \rceil_X$. Then, as $p_2(X)$ is a configuration that contains b^+, by definition of copycat (and as the projections must be locally injective), there is a unique element of the form (e, b^-) in X. Let us define $\theta(\lceil (\star, b^+) \rceil_X) = e$. Let us show that this is well defined. If we have Y such that $\lceil (\star, b^+) \rceil_X = \lceil (\star, b^+) \rceil_Y$, then we have a unique element of the form (e', b^-) in Y. But that implies that $(e', b^-) \in \lceil (\star, b^+) \rceil_Y = \lceil (\star, b^+) \rceil_X$, thus $e = e'$.
Fourthly, let us consider an element of the form $\lceil (\star, b^-) \rceil_X$.

Lemma 3.3.18 :
There exists $Y \in \mathcal{C}^o(P)$ such that $X \subseteq Y$ and $\lceil b^- \rceil \subseteq g \circ p_1(Y)$.

Δ. Let us suppose that $a \in \lceil b^- \rceil$ is minimal such we have not constructed a $Y \in \mathcal{C}^o(P)$ with $X \subseteq Y$ and $\lceil a \rceil \subseteq g \circ p_1(Y)$. As we have taken a to be minimal and as $\lceil \rceil a]_X$ is finite, we can consider that we have Y' such that $X \subseteq Y'$ and $\lfloor a^- \rfloor \subseteq g \circ p_1(Y')$.
If a is positive, then in C_B we have $a^- \leqslant a^+ \leqslant b^- \leqslant b^+$ where a^- and b^+ are in the dual (and thus sent to the left to a^+ and b^- respectively). But then $a^- \in p_2(X)$ and thus $a^+ \in g \circ p_1(X)$. Thus $Y := Y'$ verifies the required property.

If a is negative. Then, if $a^- \in g \circ p_1(Y')$, let $Y := Y'$. Else $g \circ p_1(Y') \cup \{a\}$ is a negative extension of $g \circ p_1(Y')$ and thus, as s is negatively saturated, there exists $Z \in \mathcal{C}^o(S)$ such that $p_1(Y') \subseteq Z$ and $g(Z) = g \circ p_1(Y') \cup \{a\}$. Let us take this Z to be minimal. As shown in the proof of lemma **(3.3.12)** $Z := p_1(Y') \cup \{e\}$ where $g(e) = a$. And thus $Y := Y' \cup \{(e, b^+)\}$ is a configuration. $\square$

Let Y be the extension of X given by lemma **(3.3.18)**. We know that there is an element of the form (e, b^+) in it and thus we define $\theta(\lceil(\star, b^-)\rceil_X) = e$. This
To show that this e is unique, let us consider two configurations X_1 and Y_1 such that $\lceil(\star, b^-)\rceil_{X_1} = \lceil(\star, b^-)\rceil_{X_2}$, then let $X = \lceil(\star, b^-)\rceil_{X_1}$, it is also a configuration and it is clear that $\lceil(\star, b^-)\rceil_X = \lceil(\star, b^-)\rceil_{X_1}$. Let us now consider two configurations Y_1 and Y_2 that verify the lemma with respect to X_1 and X_2 respectively.
Then let us show that $Y_3 = Y_1 \cap Y_2$ verifies the lemma with respect to X. First of all, it is clear that Y_3 is a configuration that contains X. Let us now suppose that we have $a \in \lceil b^- \rceil$ minimal such that $a \notin g \circ p_1(Y_3)$. If a is positive, we have shown that there is an element (e, a^-) in X and thus in Y_3. Therefore a must be negative. But we know that there exists $e_i \in p_1(Y_i)$ such that $g(e_i) = a$ for $i = 1, 2$.
Let e' be a direct predecessor of e_1, then as s is innocent, $g(e') \downarrow$ and $g(e') \prec a$. But then, as a is minimal, we have $e' \in Y_3 \subseteq Y_2$. Moreover, there exists $e'' \leqslant e_2$ such that $g(e'') = g(e')$, but as g is injective on Y_2, we must have $e' = e'' \leqslant e_2$. We symmetrically that all direct predecessors of e_2 are predecessors of e_1 and thus we know that e_1 and e_2 have the same strict history. Thus they are coincident. As s is negative coincidence free, $e_1 = e_2$. Therefore $a \in g \circ p_1(Y_3)$, but that is absurd. Thus Y_3 verifies the required property.
Let us now suppose that we have constructed $\theta(\lceil(\star, b^-)\rceil_X) = e_1$ using Y_1 and $\theta(\lceil(\star, b^-)\rceil_X) = e_2$ using Y_2. Then we can use $Y_1 \cap Y_2$ to construct yet another possible image e_3. But e_1 and e_3 are both in Y_1 and have the same image by p_2 thus there are equal. Symmetrically, $e_2 = e_3$, and thus θ is well defined.
Let us now show that θ is a map of event structures. Let $X \in \mathcal{C}^o(P)$, then by iterating lemma **(3.3.18)**, on all elements of the form $\lceil(\star, b^-)\rceil_X$ we obtain a configuration X' such that

$$\{\lceil(\theta(z), b^+)\rceil_X \mid (\star, b^-) \in X \wedge z = \lceil(\star, b^-)\rceil_X\} \subseteq X'$$

We can consider this X' to be minimal. Let X'' be $X' \setminus \{(e, b^-) \in X' \mid (\star, b^+) \notin X'\}$. Then X'' is a configuration. Indeed, it is consistent as the subset of a consistent one. Let us show it is down-closed. Let $z \in X''$ and $y \leqslant_{X'} z$. Then, as $z \in X'$ and X' down-closed, we $y \in X'$. If it is not of the form (e, b^-) it is still in X''. Let us now suppose it is of this form, and that it is maximal such that it is not in X''.
Let z' be such that $(e, b^-) \prec z' \leqslant z$. Lemma **(1.4.5)** implies that there exists i such that $p_i(e, b^-) \prec p_i(z')$. First let us suppose $i = 1$. As e must be positive and s is innocent, $g(e) \prec g \circ p_1(z')$. Moreover, for the square to commute, we must have $l \circ p_2(z') \downarrow$ and thus $p_2(z') \downarrow$. Therefore b^- and $p_2(z')$ are both in $B^\perp$ and such that $l(b^-) \leqslant l \circ p_2(z')$. By the definition of copycat, we must have $b^- \leqslant p_2(z')$. If $p_2(z')$ is positive, then $b^+ \leqslant p_2(z')^- \prec p_2(z')^+$ and thus X' must contain $(\star, b^+)$. If $p_2(z')$ is negative, then as we have chosen (e, b^-) maximal such that it is in X' but not in X'', z' must be in X'' and thus $(\star, p_2(z')^+) \in X'$. Thus $(\star, b^+) \in X'$.
Let us now suppose that $i = 2$. Then we have $b^- \prec p_2(z')$. If $p_2(z) \in B$, the definition of copycat implies that $p_2(z) = b^+$ and thus $(\star, b^-) \in X$. If $p_2(z) \in B^\perp$ we are back in a case we have considered before, and thus $(\star, b^+) \in X'$. In all cases X'' contains (e, b^-), and so that is absurd.
Thus we have shown that X'' is a configuration. Moreover $\theta(X) = p_1(X'')$. As p_1 is a map of event structures, $p_1(X'') \in \mathcal{C}^o(S)$, and thus $\theta(X) \in \mathcal{C}^o(S)$.
A rapid look at its construction shows that θ is strictly polarised. Moreover let us show it is rigid. First, let us remark that l is an extremal epimorphism and thus according to lemma **(3.3.5)**, its pullback p_1 is extremal too. In particular, lemma **(3.3.3)** implies it is rigid. Let x_1 and $x_2 \in P$ such that $x_1 \leqslant x_2$ and $x_i \downarrow$ for $i = 1, 2$ (and thus they contain a $\star$). Let us show by induction on the number of events between x_1 and x_2 that $\theta(x_1) \leqslant \theta(x_2)$. If we have x_3 such that $x_1 < x_3 < x_2$ and $\theta(x_3) \downarrow$ we can conclude

immediately by induction. One other simple case is if $x_i = (e_i, \star)$ for $i = 1, 2$, but then, as p_1 is rigid, we have $\theta(x_1) = p_1(x_1) \leqslant p_2(x_1) = \theta(x_2)$.
Now, if we have $x_1 = (\star, b_1) \prec (\star, b_2) = x_2$, lemma (**1.4.5**) implies that $b_1 \prec b_2$ in C_B, but, as the square commutes, they must both be in $\mathfrak{D}_r = B$, we can conclude that $b_1 \prec b_2$ in B. And thus $g \circ \theta(x_1) \prec g \circ \theta(x_2)$. Therefore there exists $e \leqslant \theta(x_2)$ such that $g(e) = g \circ \theta(x_1)$. But then, as $theta\lceil x_2 \rceil$ is a configuration that contains $\theta(x_1)$ and $\theta(x_2)$ (and thus e), local injectivity implies that $e = \theta(x_1)$ and thus that $\theta(x_1) \leqslant \theta(x_2)$.
One may remark that lemma 1.4.5 implies that we cannot have $x_1 = (\star, b_1) \prec (e_2, \star) = x_2$ (or vice versa). Thus the last case is when x_1 is not direct predecessor to x_2 and they are only separated by elements not in $\mathfrak{D}_\theta$ (and thus of the form (e, b)). We will only consider the case where $x_i = (\star, b_i)$, as the two other cases are treated in a similar manner. Then, there exists $y_1 = (e_1, b'_1)$, $y_2 = (s_2, b'_2) \in P$ such that $x_1 \prec y_1 \leqslant y_2 \prec x_2$. But then lemma (**1.4.5**) implies that $b_1 \prec b'_1$ and $b'_2 \prec b_2$. Moreover, for the square to commute, for $i = 1, 2$, we must have $r(b_i)\downarrow$ and $l(b'_i)\downarrow$ and thus the definition of copycat implies that $b'_i = b_i^\perp$. Thus $\theta x_i = e_i$ and as p_1 is rigid, $\theta(x_1) \leqslant \theta(x_2)$.
Let us now show that it is surjective on configurations. Let $X \in \mathcal{C}^o(S)$ be a configuration. As p_1 is extremal, lemma (**3.3.3**) implies that it is surjective on configurations. Let $Y \in \mathcal{C}^o(P)$ such that $p_1(Y) = X$. Let $Y' = Y \cup \{(\star, b^+) \mid (e, b^-) \in Y\}$. Then it is straightforward to check that $Y' \in \mathcal{C}^o(P)$. Moreover, let $Y'' = Y' \backslash \{(\star, b^-) \mid (e, b^+) \notin Y'\}$. Then $Y'' \in \mathcal{C}^o(P)$. Indeed, it is consistent. Now let us consider $z \in Y''$ and $(\star, b^-) \leqslant_{Y'} z$ and let us show by induction on the number of events between $(\star, b^-)$ and z, that $(\star, b^-) \in Y''$. Let $z'' \in P$ such that $(\star, b^-) \prec z' \leqslant z''$. If $z' = (e, b')$, lemma (**1.4.5**) implies that $b^- \prec b'$ and as $r(b^-)\downarrow$ and $l(b')\downarrow$ for the square to commute, we must have $b' = b^+$ and thus $(e, b^+) = z' \in Y''$, therefore $(\star, b^-) \in Y''$. If $z' = (\star, b')$, then $b^- \prec b'$ and $r(b')\downarrow$ thus the definition of copycat implies that b' is negative. Thus by induction $z' \in Y''$ and thus $(e', b'^+) \in Y'$, thus $(e, b^+) \in Y'$. Therefore $(\star, b^-) \in Y''$.
Thus we have proved that $Y'' \in \mathcal{C}^o(P)$, and as $\theta(Y'') = p_1(Y) = X$, θ is indeed surjective on configurations. Therefore according to lemma (**3.3.3**), it is a strong epimorphism.
Furthermore θ is injective on finite configurations of P that only have elements of $\mathfrak{D}_\theta$ as their top elements, i.e. configurations of visible events. Let X and Y be two such configurations such that $\theta(X) = \theta(Y)$. Let us show that $X \subset Y$. Let $x \in X$. If $\theta(x)\downarrow$, then an analysis of all four cases shows that a given $e \in S$ can only be the image by θ of a specific pair (notwithstanding the configuration), and thus $x \in Y$. There remains the case $x = (e, b)$. But as X has all top elements in $\mathfrak{D}_\theta$, there exists $y \in \mathfrak{D}_{theta}$ such that $y \in X$ (and thus in Y, as we have seen previously) and $x \leqslant y$.
If $y = (e', \star)$, then as p_1 is rigid, $e \leqslant e'$ and thus there exists an element of the form $(e, b') \in Y$, but then $b'^\perp = p_2 \circ l(e, b') = p_1 \circ r(e) = p_2 \circ l(e, b) = b^\perp$ thus $b = b'$ and $x \in Y$. Now, if $y = (\star, b')$, let us suppose it is minimal such that $x \leqslant y$ and $\theta(y)\downarrow$. Then there exists (e'', b'') such that $(e, b \leqslant (e'', b'') \prec (\star, b'))$. As shown earlier, we must thus have b' positive and $b'' = b'^-$. Therefore there is an element of the form $(e', b'^-) \in Y$. But then $e' = \theta(\lceil(\star, b')\rceil_Y)$ and $e'' = \theta(\lceil(\star, b')\rceil_X)$ are both in $\theta(X) = \theta(Y)$. But as we have already seen, a given image can come only from one pair, and thus we must have $e' = e''$. We can then conclude as in the previous case to show that $x \in Y$. The symmetry of the problem allows us to say that we have $Y \subset X$ too, and thus they are equal.
Thus, lemma (**3.3.4**) implies that $\mathcal{V}\theta$ is a monomorphism. Moreover lemma (**3.3.6**) implies that it is also a extremal map, thus, according to lemma (**2.2.4**), it is an isomorphism between $\mathcal{V}(s \circledast \mathfrak{cc})$ and $\mathcal{V}s$. But we have showed in the proof of lemma (**3.3.2**), that the visible part of a fully visible span is itself and thus we have the required isomorphism.
To prove the reciprocal one just has to consider counter-examples that are relatively easy to find but whose description can be long... □

Moreover, I believe that when they exist, the isomorphisms are natural and make the triangle of the bicaterical axioms commute. The proofs are terribly tedious... Nevertheless, if we admit this little bit, we can at last define the category of spans that we have been looking for.

Definition 3.3.19 ($\mathcal{S}pan_{Inn}$) **:**
Negatively saturated, negative coincidence free, innocent and fully visible polarised spans form a bicategory $\mathcal{S}pan_{Inn}$ *whose composition is* $\circledast$ *and whose identities are copycat spans.*

Δ. Associativity is inherited from $\mathcal{VS}pan^{\pm}_{\oplus}$. The unitors are given by lemma (**3.3.17**) □

Chapter 4

Some monads and adjunctions

4.1 An adjunction for rigidity

The first construction, we are to consider is how to represent non rigid maps out of rigid maps. To do this we define the augmentations of an event structure and show that a partial rigid map into the augmentations is exactly a partial map into the event structure.

Definition 4.1.1 (Augmentations) **:**
Let A be an event structure. We define an A-augmentation to be a pair (X, α) where $X \in \mathcal{C}(A)$ and α is a finitary order on X that refines $\leqslant_A$, i.e. for all a, $b \in X$ such that $a \leqslant_A b$ then $a\alpha b$.
A-Augmentations can be ordered by taking $(X, \alpha) \leqslant (Y, \beta)$ if and only if $X \subseteq Y$ and the inclusion from (X, α) to (Y, β), viewed as elementary event structures, is a rigid map of event structures.

Lemma 4.1.2 :
Let A be an event structure. The augmentations of A form a finitary prime algebraic domain whose primes are the finite augmentations with a top element.

Δ**.** The A-augmentations are consistent complete, indeed, let X be a set of A-augmentations, then its least upper bound is the A-augmentation $\bigcup X$ with the order inherited by the A-augmentations in X. This order is well defined because let (Y_1, α_1) and (Y_2, α_2) be two elements of X, and a, $b \in Y_1 \cap Y_2$ such that $a\alpha_1 b$. Then as X is finitely bounded, there exists (Z, β) such that the inclusion of (Y_i, α_i) into (Z, β) is rigid, for $i = 1, 2$. Thus as $a\alpha_1 b$ we also have $a\beta b$ and thus $a\alpha_2 b$.
Moreover $\bigcup X$ is down-closed as it is the union of down-closed sets and it is consistent as, if there is $Y \subseteq_{\mathrm{f}} \bigcup X$ then there is a covering of Y with a finite number of $x \in X$ which are bounded as X is consistent complete and thus Y is a subset of the bound (that is itself a configuration). It is also clear that the order we have defined refines the order of A. Thus it is a A-augmentation.
Moreover, let $p := (P, \theta)$ be a A-augmentation with a top element e, and X a set of A-augmentations such that $p \leqslant \coprod X := (Z, \beta)$. Then there is a $(Y, alpha) \in X$ such that $e \in Y$. But then $y\theta e$ implies $y\beta e$ and thus y must be in Y too and we have$y\alpha e$. Therefore $p \leqslant (Y, alpha)$ and p is indeed a complete prime.
Finally, let $x = (X, \alpha)$ be a A-augmentation, then $x = \coprod\{\lceil e \rceil_a lpha \mid e \in X\}$. □

Let A be an event structure, we define $\mathrm{Aug}(A)$ to be the event structure associated with the finitary prime algebraic domain of the A-augmentations.

Theorem 4.1.3 :
The inclusion functor from $\mathcal{E}_{pr}$ to $\mathcal{E}_p$ has a right adjoint whose object function is Aug.

Δ**.** Let us first show that the map ε_A from $\mathrm{Aug}(A)$ to A that takes an augmentation with a top to its top is a map of $\mathcal{E}_p$.
Let $X \in \mathcal{C}^o(\mathrm{Aug}\, A)$, As X is a configuration, it is bounded in the A-augmentations, thus there exists (Y, α) such that all element of X are included rigidly into (Y, α). Thus $X = \{\lceil x \rceil_\alpha \mid x \in \bigcup X\}$. Then $\varepsilon_A(X) = \bigcup X \subseteq Y$ is a configuration. Moreover, ε_A is clearly injective on X.
Let us now show that it is universal from Incl to A. Let $f : B \rightharpoonup A$ in $\mathcal{E}_p$. Let $b \in B$, such that $f(b)\downarrow$. Then the configuration $f(\lceil b \rceil)$ inherits an order from B by taking $f(x) \leqslant_f f(y)$ if and only if $x \leqslant_B y$ (as f is injective on configuration, this order is well-defined). Lemma (**1.1.4**) states that f reflects order on

configurations and thus this order refines the order in A. Therefore $(f(\lceil b\rceil), \leqslant_f)$ is an A-augmentation. Moreover, $f(b)$ is its top element, and thus it is an event of $\mathrm{Aug}(A)$.
We can therefore consider the map $\overline{f}$ from B to $\mathrm{Aug}(A)$ that takes a b, when $f(b)\downarrow$, to $(f(\lceil b\rceil) \leqslant_f)$ and that is undefined elsewhere. Let us show that this map is a map of $\mathcal{E}_{pr}$. Let $X \in \mathcal{C}^o(B)$ then for all $x \in X$, $\overline{f}(x)$ (if defined) can be injected rigidly into $(f(X), \leqslant_f)$ and thus $\overline{f}(X)$ is consistent. Moreover, let $y \in \overline{f}(X)$ and $x \leqslant y$. Then there exists $y' \in X$ such that $y = \lceil f(y')\rceil_{\leqslant_f}$. But then, as x has a top element and can be injected rigidly into y, there exist $x' \in \lceil f(y')\rceil \subseteq \lceil f(X)\rceil \subseteq f(\lceil X\rceil) = f(X)$ such that $x = \lceil x'\rceil_{\leqslant_f}$. But then there exist $x'' \in X$ such that $x' = f(x'')$ and thus $x = \overline{f}(x'')$.
It remains to show that $\overline{f}$ is locally injective. But if $\overline{f}(x_1) = \overline{f}(x_2)$, they must have the same top element and thus $f(x_1) = f(x_2)$. As f is injective on configurations we can conclude.
It is evident that $\varepsilon_A(\overline{f}(b)) = f(b)$. Moreover, let $h : B \rightharpoonup \mathrm{Aug}(A)$ be a map in $\mathcal{E}_{pr}$ that verifies this equality. Let $b \in B$, if $f(b)\uparrow$, then as ε_A is total, we must have $h(b)\uparrow$ and $\overline{f}(b)\uparrow$.
Let us now suppose that $f(b)\downarrow$, and let $h(b) = (X, \alpha)$. Let us first show that $\lceil h(b)\rceil = \{\lceil x\rceil_\alpha \mid x \in X\}$.Let $y \in \lceil h(b)\rceil$, then as $y \leqslant h(b)$, it is a down-closed subset of X inheriting its order from α. Moreover as it is in $\mathrm{Aug}(A)$, it has a top and thus is of the form $\lceil x\rceil_\alpha$ for some $x \in X$. Conversely, it is evident that $\lceil x\rceil_\alpha$ can be injected rigidly into (X, α).
Let us now show that $X = f(\lceil b\rceil)$. Let $x \in X$, as shown before this is equivalent to $\lceil x\rceil_\alpha \in \lceil h(b)\rceil$. But as h is rigid, according to lemma (**1.1.6**), $\lceil h(b)\rceil = h(\lceil b\rceil)$. Thus this is equivalent to the existence of $y \in \lceil b\rceil$ such that $h(y) = \lceil x\rceil_\alpha$. But then $f(y) = \varepsilon_A(h(y)) = x$ and thus $x \in f\lceil b\rceil$.
Let us finally show that $\alpha = \leqslant_f$. Let x, $y \in X$. As shown before, there exist x' and $y' \in \lceil b\rceil$ such that $f(x') = x$, $f(y') = y$, $h(x') = \lceil x\rceil_\alpha$ and $h(y') = \lceil y\rceil_\alpha$. But then $x\alpha y$ is equivalent to the fact that $\lceil x\rceil_\alpha$ can be injected rigidly into $\lceil y\rceil_\alpha$ and thus to $h(x') \leqslant h(y')$. As h is rigid, that is equivalent to $x' \leqslant_B y'$ and thus we have proved that $f(x')\alpha f(y')$ is equivalent to $x' \leqslant_B y'$.
Therefore, if $f(b)\downarrow$ then $h(b) = \overline{f}(b)$. As we have shown that $f(b)\uparrow$ implies both $\overline{f}(b)\uparrow$ and $h(b)\uparrow$, we have indeed proven that $h = \overline{f}$. □

This process can be copied to obtain a right adjoint to the inclusion functor from $\mathcal{Fil}_{pr}$ into $\mathcal{Fil}_p$.

Definition 4.1.4 (Filiform augmentations) **:**
Let A be a filiform event structure. We define an filiform A-augmentation to be a pair (X, α) where $X \in \mathcal{C}(A)$ and α is a filiform finitary order on X that refines $\leqslant_A$, i.e. for all a, $b \in X$ such that $a \leqslant_A b$ then $a\alpha b$.
Filiform A-augmentations can be ordered by saying that $(X, \alpha) \leqslant (Y, \beta)$ if and only if $X \subseteq Y$ and the inclusion from (X, α) to (Y, β) viewed as elementary event structures is a rigid map of event structures.

Lemma 4.1.5 :
Let A be an event structure. The filiform augmentations of A form a finitary prime algebraic domain whose primes are the finite augmentations with a top element.

Δ. The one difference with the preceding proof is to show that the least upper bound we have defined is a filiform augmentation. Indeed, let X be a set of filiform A-augmentation and let $(\bigcup X, \beta)$ be the least upper bound we have already defined. Let us show it is filiform.
Let y, z_1, $z_2 \in \bigcup X$ such that z_1, $z_2\beta y$. Then there is a $(Y, alpha) \in X$ such that $y \in Y$. But then $(Y, alpha) \leqslant (\bigcup X, \beta)$ and thus we must also have z_1, $z_2 \in Y$ and z_1, $z_2\alpha y$. As $(Y, alpha)$ is filiform this implies that $z_1\alpha z_2$ or $z_2\alpha z_1$ and therefore $z_1\beta z_2$ or $z_2\beta z_1$. □

Let A be a filiform event structure, we define $\mathrm{Filaug}(A)$ to be the event structure associated with the finitary prime algebraic domain of the filiform A-augmentations.

Lemma 4.1.6 :
Let A be a filiform event structure, then $\mathrm{Filaug}(A)$ is filiform.

Δ. Let (X, α), Y, Z be three filiform A-augmentations with a top element such that Y, $Z \leqslant (X, \alpha)$. Then there exists y, $z \in X$ such that $Y = \lceil y \rceil_\alpha$ and $Z = \lceil z \rceil_\alpha$. But then y and z are both smaller that the top element of (X, α) which is a filiform order and thus we can suppose $y \alpha z$. But then that implies that $\lceil y \rceil_\alpha$ is included rigidly into $\lceil z \rceil_\alpha$ and thus that $Y \leqslant Z$. □

Theorem 4.1.7 :

The inclusion functor from $\mathcal{Fil}_{pr}$ to $\mathcal{Fil}_p$ has a right adjoint whose object function is Filaug.

Δ. The proof is essentially the same that the one of theorem (**4.1.3**). The only addition is to show that $(f(\lceil b \rceil), \leqslant_f)$ is filiform. So let $f(x)$, $f(y)$ and $f(z) \in f(\lceil b \rceil)$, such that $f(y)$, $f(z) \leqslant_f f(x)$. Then y, $z \leqslant x$ and thus we can suppose $y \leqslant z$. But then that implies that $f(y) \leqslant_f f(z)$. □

4.2 Filiform event structures and rigidity

Let us now consider a rapid result, that in itself will not be useful but is worth noticing.

Lemma 4.2.1 :

Let A be a filiform event structure, B an event structure and $f : A \rightharpoonup B$ rigid. Then the image of f in B is a filiform event structure.

Δ. Let y_1, y_2 and $f(z)$ in $\Im f$ such that y_1, $y_2 \leqslant f(z)$. Then there exists x_1, $x_2 \leqslant z$ such that $f(x_1) = y_1$ and $f(x_2) = y_2$. As A is filiform, we can suppose $x_1 \leqslant x_2$ and thus as f is rigid, $y_1 = f(x_1) \leqslant f(x_2) = y_2$.□

Lemma 4.2.2 :

Let A be an event structure and let $(F_i)_{i \in I}$ be a family of filiform down-closed sub event structures. Then $\bigcup_{i \in I} F_i$ is filiform and down-closed.

Δ. let $x \in \bigcup_{i \in I} F_i$ and y_1, $y_2 \leqslant x$. Let $i \in I$ such that $x \in F_i$, then as F_i is down-closed, y_1, $y_2 \in F_i$ and, as it is filiform, y_1 and y_2 are comparable. Thus $\bigcup_{i \in I} F_i$ is filiform.
Moreover $y_1 \in F_i \subseteq \bigcup_{i \in I} F_i$, thus it is down-closed. □

Definition 4.2.3 (Filiform component) **:**
Let A be an event structure and let $\mathrm{Mfil}(A)$ *be its maximal down-closed filiform subset. It exists in virtue of lemma (4.2.2) as the union of all the down-closed filiform subsets of A.*

Theorem 4.2.4 :

The inclusion functor from $\mathcal{E}_{pr}$ to $\mathcal{Fil}_{pr}$ has a right adjoint of which Mfil is the object function. As the inclusion functor is full and faithfull, it is a coreflection.

Δ. It suffices to show that the inclusion map ε_A from $\mathrm{Incl} \circ \mathrm{Mfil}(A)$ to A is a universal arrow from Incl to A. Let B be a filiform event structure and let $f : \mathrm{Incl}(B) \rightharpoonup A$. Then lemma (**4.2.1**) implies that $\Im f$ is filiform and included in A, thus it is in $\mathrm{Mfil}(A)$. Thus f can be seen as a map from B to $\mathrm{Mfil}(A)$ such that $\varepsilon_A \circ \mathrm{Incl}(f) = f$.
Moreover if there is a another map $h : B \rightharpoonup \mathrm{Mfil}(A)$ such that $\varepsilon_A \circ \mathrm{Incl}(h) = f$, then as ε_A is monic, $\mathrm{Incl}(h) = \mathrm{Incl}(f)$ and thus $f = h$. □

4.3 A copying mechanism

To give the notion of copying in an event structure a good enough sense, we have to consider a notion of symmetry to be able to, in a way, identify the copies of a given element. We will define it here without the symmetry, but then cannot show it is a monad.

The copying we consider here is quite complicated as it recursively copies events from the bottom to the top, and does not simply copy the event structure itself.

Definition 4.3.1 ($!E$) **:**
Let E be an event structure and let us define $!E$ to be the smallest set such that $(P, i, e) \in !E$ if and only if $P \subseteq !E$, $i \in \mathbb{N}$, $e \in E$, ε_E is a bijection from P to $\lfloor e \rfloor$, where $\varepsilon_E(P', i', e') = e'$, and P is transitive, i.e. if $(P_1, i_1, e_1) \in P$ and $(P_2, i_2, e_2) \in P_1$ then $(P_2, i_2, e_2) \in P$.
Let $X \in \mathfrak{C}_{!E}$ if and only if $X \subseteq_{\mathrm{f}} !E$ and for all $Y \subseteq X$ such that $\varepsilon_{E \restriction Y}$ is injective, $\varepsilon_E(Y) \in \mathfrak{C}_E$. And let $(P', i', e') <_{!E} (P, i, e)$ of and only if $(P', i', e') \in P$.

Lemma 4.3.2 :
$!E$ is an event structure. Moreover, if E is filiform, $!E$ is filiform too.

Δ. Let us first check that it is a event structure.
Firstly, let us check that we have indeed defined a strict order. We cannot have $(P, i, e) <_{!E} (P, i, e)$ because that would imply that $(P, i, e) \in P$ and thus that $e = \varepsilon_E(P, i, e) \in \varepsilon_E(P) = \lfloor e \rfloor$, which is absurd. Moreover, let $(P_1, i_1, e_1) <_{!E} (P_2, i_2, e_2) <_{!E} (P_3, i_3, e_3)$. Then $(P_1, i_1, e_1) \in P_2$ and $(P_2, i_2, e_2) \in P_1$, as P_1 is transitive, we have $(P_1, i_1, e_1) \in P_3$ and thus $(P_1, i_1, e_1) <_{!E} (P_3, i_3, e_3)$.
Secondly, let $(P, i, e) \in !E$, then $\#(\lceil (P, i, e) \rceil) = \#(P) + 1$, and as P is injected into $\lfloor e \rfloor$ that is finite, P is finite. Thus the order is finitary.
Thirdly, $\varepsilon_E(\{(P, i, e)\}) = \{e\} \in \mathfrak{C}_E$. Thus singletons are consistent.
Fourthly, if $Y \subseteq X$ then for all $Z \subseteq Y$ such that $\varepsilon_{E \restriction Z}$ is injective, we also have $Z \subseteq Y$ and thus if X is consistent, $\varepsilon_E(Z) \in \mathcal{C}^o(E)$ and therefore Y is consistent.
Finally, let $(P, i, e) \in X \in \mathfrak{C}_{!E}$ and let $(P', i', e') <_{!E} (P, i, e)$. Then $(P', i', e') \in P$ and thus $e' = \varepsilon_E(P', i', e') \in \varepsilon_E(P) = \lfloor e \rfloor$. Let $Z \subseteq X \cup \{(P', i', e')\}$ such that $\varepsilon_{E \restriction Z}$ is injective. If $Z \subseteq X$ then we know that $\varepsilon_E(Z) \in \mathcal{C}^o(E)$, else $Y = \varepsilon_E(Y \setminus \{(P', i', e')\}) \in \mathcal{C}^o(E)$ and $\varepsilon_E(Z) = Y \cup e'$. Moreover $e \in Y$ and thus $Y \cup e' \in \mathcal{C}^o(E)$. Therefore $X \cup \{(P', i', e') \in \mathcal{C}^o(!E)$.
Furthermore, let us show that if E is filiform then $!E$ is filiform.
Let (P_1, i_1, e_1), $(P_2, i_2, e_2) \leqslant_{!E} (P, i, e)$, the case when there is at least one equality is evident, thus we can suppose that they are strictly inferior, i.e. $(P_1, i_1, e_1), (P_2, i_2, e_2) \in P$, and different. But then $\varepsilon_E(P_j, i_j, e_j) = e_j \in \lfloor e \rfloor$ for $j = 1, 2$ and thus as E is filiform, e_1 and e_2 are comparable. Let us suppose that $e_1 \leqslant_E e_2$. But as (P_1, i_1, e_1) and (P_2, i_2, e_2) are distinct and that ε_E is injective on P_2, it follows that $e_1 < e_2$ and thus that there exists $(P_1', i_1', e_1) \in P_2$. But, as P is transitive, that implies that $(P_1', i_1', e_1) \in P$ and, as $\varepsilon_{E \restriction P_3}$ is injective, it implies that $P_1' = P_1$ and $i_1' = i_1$. And thus $(P_1, i_1, e_1) <_{!E} (P_2, i_2, e_2)$. □

Let E be an event structure and $x, y \in !E$. If $\varepsilon_E(x) = \varepsilon_E(y)$ they are said to be copies of one another.

Chapter 5

Games

There are many presentations of simple games, for example [Hyl97] and [HHM07], or disguised as affine sequential algorithms in [Cur94]. In my opinion, it is [HHM07], that gives the clearest, and most formal definition of all, thus I'll work with this one.
The goal in this chapter is to show that our category $\mathcal{S}pan_{Inn}$ is rich enough to represent already existing games.

5.1 Schedules and simple games

This section gives a somewhat different account of the definitions in [HHM07], the main difference is that schedules are defined in a way that I find simpler to work with.
First let us define those schedules. They are objects that allow to mix two orders. They will be used to construct the arena later on.

Definition 5.1.1 (Schedules) **:**
For all $n \in \mathbb{N}$, let $(n) := [\![1; n]\!]$ with the convention that $(0) = \emptyset$.
For all $p \in \mathbb{N}$, $q \in \mathbb{N}$, σ a schedule $\sigma : p \twoheadrightarrow q$ is a relation from (p) to (q) such that

- *(i) For all x, $y \in (p)$ and z, $t \in (q)$, $x \leqslant y\sigma z \leqslant t$ implies $x\sigma t$.*
- *(ii) If $x \in (p)$ is odd, then, for all $y \in (q)$, $x\sigma y$ implies $(x+1)\sigma y$.*
- *(iii) If $y \in (q)$ is even, then, for all $x \in (p)$, $y\sigma^* x$ implies $(y+1)\sigma^* x$.*
- *(iv) For all $x \in (p)$, $1\sigma^* x$.*

where σ^*, the symmetric relation, is the relation defined on $q \times p$ by $y\sigma^* x := \neg(x\sigma y)$.

We say that $\sigma : p \twoheadrightarrow q$ is a right schedule if $p\sigma q$ and that it is a left schedule if $q\sigma^\perp p$ (with the obvious conventions that schedules from 0 are all right schedules).
As $p\sigma q$ and $q\sigma^\perp p$ are mutually exclusive, a schedule cannot be both right and left.
If we take the convention that one may only write $x\sigma y$ (be it true or false) if $x \leqslant p$ and $y \leqslant q$, then the last condition implies that $q \geqslant 1$, the second condition that if the schedule is right then p is event and the third that if the schedule is left then q is odd.
We also define σ^- to be the schedule σ without its top element (i.e. $\sigma_{|p\times(q-1)}$ if it is right and $\sigma_{|(p-1)\times q}$ if it left).
Let us now show that the first requirement of a schedule also holds for its symmetric.

Lemma 5.1.2 :
Let $\sigma : p \twoheadrightarrow q$ be a schedule and let x, $y \in (q)$ and z, $t \in (p)$. Then $x \leqslant y\sigma^* z \leqslant t$ implies $x\sigma^* t$.

Δ. Let us suppose that $t\sigma x$, then $z \leqslant t\sigma x \leqslant y$, and thus $z\sigma y$, but that is absurd. □

In fact, a schedule only needs to order even elements, the following definition and lemma make that clear.

Definition 5.1.3 (Alternative definition of schedules) **:**
Let $(p)^e := \{x \in (p) \mid x \text{ is even}\}$.

Let $p \in \mathbb{N}$ and $q \in \mathbb{N}^*$, an even schedule $\sigma : p \multimap q$ is a relation from $(p)^e$ to $(q)^e$ such that for all $x, y \in (p)^e$ and $z, t \in (q)^e$, $x \leqslant y\sigma z \leqslant t$ implies $x\sigma t$, and such that if q is even, then p is even and $p\sigma q$.

Lemma 5.1.4 :

There is a bijective correspondence between schedules and even schedules.

Δ. If one has a schedule, one can obtain an even schedule σ^e by restricting it to even numbers. We have already noted that $q \leqslant 1$ for any schedule, and that if σ is left, then q is odd, and thus if q is even, the schedule must be right and thus the last condition holds.
If one has an even schedule $\sigma : p \multimap q$, for all $x \in (p)$ and $y \in (q)$, let $x\overline{\sigma}y$ if and only if there exists $z \in (p)^e$ and $t \in (q)^e$ such that $x \leqslant z\sigma t \leqslant y$. Let us check that $\overline{\sigma}$ is a schedule.

(i) The first condition is implied by the definition.

(ii) Let $x \in (p)$ be odd and $y \in (q)$ such that $x\overline{\sigma}y$. Then there exists $z \in (p)^e$ and $t \in (q)^e$ such that $x \leqslant z\sigma t \leqslant y$. But as x is odd $(x+1) \leqslant z$ and thus $(x+1)\overline{\sigma}y$.

(iii) Let $y \in (q)$ be even and $x \in (p)$ such that $x\overline{\sigma}(y+1)$. Then there exists $z \in (p)^e$ and $t \in (q)^e$ such that $x \leqslant z\sigma t \leqslant (y+1)$. But then, as t is even and y too, $t \leqslant y$ and thus $x\overline{\sigma}y$. Moreover, if $y+1 > q$ then, that implies that $y = q$ is even and thus p is even and $p\sigma q$. Thus $x \leqslant p\sigma q = y$.

(iv) First of all, it is indeed the case that $1 \leqslant q$. Furthermore, let $x \in (p)$ such that $x\overline{\sigma}1$ then there exists $y \in (p)^e$ and $z \in (q)^e$ such that $x \leqslant y\sigma z \leqslant 1$. But that is impossible since there are no even numbers smaller that 1 and strictly bigger than 0.

Let us now show that these two transformations are mutually reciprocal. First, let $\sigma : p \multimap q$ be an even schedule.
Let $x \in (p)^e$ and $y \in (q)^e$ such that $x\overline{\sigma}^e y$, then there exists $z \in (p)^e$ and $t \in (q)^e$ such that $x \leqslant z\sigma t \leqslant y$ and thus $x\sigma y$.
Let us now suppose that $x\sigma y$, then as $x \leqslant x\sigma y \leqslant y$, we do have $x\overline{\sigma}^e y$.
Secondly, let σ be a schedule, and let $x \in (p)$ and $y \in (q)$ such that $x\overline{\sigma^e}y$. Then there exists $z \in (p)^e$ and $t \in (q)^e$ such that $x \leqslant z\sigma^e t \leqslant y$ and thus $x\sigma y$.
Now, let us suppose that $x\sigma y$. If x is even, let $x' = x$ and if it is odd, let $x' = x+1$. If y is even, let $y' = y$ and if it is odd, let $y' = y-1$. In both cases $x \leqslant x'\sigma^e y' \leqslant y$ and thus $x\overline{\sigma}y$. □

Schedules form a category, whose identity and composition we are about to present.

Definition 5.1.5 (Copy-cat) **:**
For all $p \in N^*$, copy-cat is the even schedule $\mathfrak{c} : p \multimap p$ such that $x\,\mathfrak{c}\,y$ if and only if $x \leqslant y$.

Definition 5.1.6 (Composition of schedules) **:**
Let $\sigma : p \multimap q$ and $\tau : q \multimap r$ be two schedules. We define $\tau \circ \sigma : p \multimap r$ be the relational composition of the two schedules.

Δ. Let us check that $\tau \circ \sigma$ is a schedule.

(i) Let $x, y \in (p)$ and $z, t \in (r)$ such that $x \leqslant y(\tau \circ \sigma)z \leqslant t$. Then there exists $w \in (q)$ such that $y\sigma w\tau z$. And thus as σ and τ are schedules, $x\sigma w$ and $w\tau t$ and thus $x(\tau \circ \sigma)t$.

(ii) Let $x(p)$ be odd and such that $x < p$ and let $y \in (r)$ such that $x(\tau \circ \sigma)y$. Then there exists $z \in (q)$ such that $x\sigma z\tau y$. Therefore $(x+1)\sigma z$ and thus $(x+1)(\tau \circ \sigma)y$.

(iii) Let $y \in (r)$ be even such that $y < r$. Then let $x \in (p)$ and $z \in (q)$ such that $x\sigma z\tau(y+1)$. Then $z\tau y$ and thus $x(\tau \circ \sigma)y$. Thus $x(\tau \circ \sigma)^* y$ implies $x(\tau \circ \sigma)^*(y+1)$.

(iv) let $x \in (p)$, $z \in (q)$ such that $x\sigma z\tau 1$. But that contradicts the fact that for all $z \in (q)$, $1\sigma^* z$. Thus $1(\tau \circ \sigma)^* x$. □

Even schedules compose like relations. This composition correspond to the composition of schedules. This is due to the fact that if we have $x\tau y\sigma z$ with y odd, then $x\tau(y-1)\sigma z$ also holds.

Definition 5.1.7 (Υ) **:**
Let Υ be the category where objects are all (n) for $n \in \mathbb{N}$ and the arrows are schedules. They compose according to the composition just defined and the identities are the copy-cat schedules.

Δ. The composition is inherited from the relations and thus is associative. We just need to show that the copy-cat schedules are the identities. But this is clear using the even schedule definition. □

Let us now define an order on schedules.

Definition 5.1.8 (Schedule restriction) **:**
Let $\sigma : p \multimap q$ be a schedule and let $x \in (p)$ and $y := \max\{z \in (q) \mid z\sigma^\perp x\}$. Then $\sigma_{\restriction x} : x \multimap y$ is the restriction of σ to $(x) \times (y)$.
Let $y \in (q)$ and $x := \max\{z \in (p) \mid z\sigma y\}$. Then $\sigma^{\restriction y} : x \multimap y$ is the restriction of σ to $(x) \times (y)$.
We write $\sigma \prec \tau$ if there exists x such that $\sigma = \tau_{\restriction x}$ or $\sigma = \tau^{\restriction x}$.

Let us now move on to defining simple games and their strategies.

Definition 5.1.9 (Games) **:**
A game A is given by family of sets $(A(n))_{n\in\mathbb{N}^*}$ and functions $\pi_n : A(n+1) \to A(n)$.

We will hereafter take the convention that $A(0)$ is the singleton $\{\star\}$ and thus that π_0 is the constant function from $A(1)$ to $\{\star\}$.
The move in $A(2n)$ are said to be player moves, whereas those in $A(2n+1)$ are said to be opponent moves.

Definition 5.1.10 (Strategies) **:**
Let A be a game, a strategy s in A is a family of sets $s(n) \subseteq A(n)$ such that

(i) if $x \in s(n+1)$ then $\pi(x) \in s(n)$.

(ii) if $x, y \in s(2n)$ with $\pi(x) = \pi(y)$ then $x = y$.

(iii) if $x \in A(2n+1)$ such that $\pi(x) \in s(2n)$ then $x \in s(2n+1)$.

A strategy therefore is a down-closed collection of moves that verify that the player only answers one way to a certain opponent move. The last requirement is just there for technical reasons, it just states that all opponent moves that are accessible have to be present in the strategy.
To define a category of games, we need to define a function space that will be a game mixing the domain and the co-domain. Then a morphism will be a strategy on this function space.

Definition 5.1.11 ($A \multimap B$) **:**
Let A and B be games. $A \multimap B$ is the game such that $(A \multimap B)(n)$ is the set of triplets (σ, a, b) such that $\sigma : p \multimap q$ is a schedule with $p + q = n$, $a \in A(p)$ and $b \in B(q)$ and π is defined by

$$\pi(\sigma, a, b) = \begin{cases} (\sigma^-, \pi(a), b) & \textit{if } \sigma \textit{ is left} \\ (\sigma^-, a, \pi(b)) & \textit{if } \sigma \textit{ is right} \end{cases}$$

Let us now define the composition and the identity of our simple games.

Definition 5.1.12 (Composition of strategies) **:**
Let A, B and C be games, s be a strategy on $A \multimap B$ and r a strategy on $B \multimap C$. Let $r \circ s$ be the strategy on $A \multimap C$ such that $(\sigma \circ \tau, a, c) \in (r \circ s)$ if and only if it exists $b \in B$ such that $(\tau, a, b) \in s$ and $(\sigma, b, c) \in r$.

Definition 5.1.13 (Copycat strategies) **:**
Let A be a game, the copycat strategy on A, $\mathfrak{c}_A$, is the strategy on $A \multimap A$ comprising all elements $(\mathfrak{c}, a, a)$ and $\pi(\mathfrak{c}, a, a)$ where $\mathfrak{c}$ is a copy-cat schedule.

Definition 5.1.14 ($\mathcal{G}$) **:**
Let $\mathcal{G}$ be the category of simple games, whose objects are games, and the maps in $\mathcal{G}(A, B)$ are the strategies in $A \multimap B$. They compose according to the previous definition, and identities are the copycat strategies.

The inclusion of strategies makes $\mathcal{G}$ a category enriched in order, and thus a 2-category.

5.2 From sequential spans to strategies

For technical reasons, we will supposing from now on that all event structures are countable.

Definition 5.2.1 (Coincidence up to copying) **:**
Let $s = (f, S, g)$ be a span. Let x, $y \in S$, they are said to be coincident up to symmetry if they have the same strict history and such that $f(x)$ and $f(y)$ are either both undefined or both defined and copies of one another (and of course the same holds for g)

Lemma 5.2.2 :
Let s_1 and s_2 be two filiform spans free of negative coincidence up to copying that can be composed, then $s_2 \boxdot s_1$ and $\mathcal{V}s_1$ are free of negative coincidence up to copying.

The proof is rather similar to that without copying, we shall therefore omit it.

Definition 5.2.3 (Sequential span) **:**
Let A and B be two alternating, filiform, elementary event structures, such that all minimal events are negatively polarised. A sequential span from A to B is a span (f, S, g) in $\mathcal{S}pan_{Inn}(!A, !B)$ that is filiform, concrete, deterministic, alternating and rigid, free of negative coincidence up to copying and such that all minimal events in S are negative.

One can remark that in alternating event structures with negative minimal events, positive events are of event height and negative ones are of odd height. Moreover minimal elements of S must be in the domain of g, as their images are minimal thus negative too.
Let $\mathcal{S}pan_{seq}$ be the subcategory of $\mathcal{S}pan_{Inn}$ containing the sequential spans.
Let us now state some fact about these sequential spans that will be useful in the coming proofs. They will be admitted.

Lemma 5.2.4 :
Let $s = (f, S, g)$ be a sequential. If $s_1, s_2 \in S$ have a common direct predecessor, then they are both in $\mathfrak{D}_f$ or both in $\mathfrak{D}_g$.
Moreover if s_1 and s_2 are not equal, they are negative.

Lemma 5.2.5 :
Let $s = (f, S, g)$ be a sequential span and x, $y \in S$ such that they have the same strict history and their images are copies, then they are equal.

Sequential spans represent simple games in $\mathcal{S}pan_{Inn}$, thus we will first show that we can transform a sequential span into a strategy, in a functorial way.

Definition 5.2.6 (G) **:**
First, let E be a filiform, elementary event structure such that all minimal events are negatively polarised and let $G(E)$ be the game such that $G(E)(n)$ is the set of events of height n, i.e. such that $\#\lceil e \rceil = n$, and such that for all non minimal event e, $\pi(e)$ is its direct predecessor.
Let us now consider a sequential span s

$$\begin{array}{ccc} & S & \\ {\scriptstyle f}\swarrow & & \searrow{\scriptstyle g} \\ !A & & !B \end{array}$$

For all $e \in S$, let $p_e := \#(f(\lceil e \rceil))$, $q_e := \#(g(\lceil e \rceil))$ and $\sigma_e : p_e \twoheadrightarrow q_e$ the schedule defined by $x \sigma y$ if and only if $f^{-1}(\varphi_{in}(x)) < g^{-1}(\varphi_{out}(y))$ where φ_f (respectively φ_g) is the order isomorphism between (p_e) and $f(\lceil e \rceil)$ (respectively (q_e) and $g(\lceil e \rceil)$). As, by definition, f and g are injective on $\lceil e \rceil$, this all makes sense. Moreover let us write $(\_,\_,a_e)$ for the top element of $f(\lceil e \rceil)$ if any (if the set is empty, $a_e = \star$) and $(\_,\_,b_e)$ for the top element of $g(\lceil e \rceil)$.
Let us now define

$$\begin{array}{ccc} S & \overset{\theta}{\to} & A \multimap B \\ e & \mapsto & (\sigma_e, a_e, b_e) \end{array}$$

and finally, for all $n \in \mathbb{N}^*$, $G(s)(n) := \{\theta(e) \mid e \in S \text{ of height } n\}$.

Theorem 5.2.7 :
G is a pseudo-functor from $\mathcal{S}pan_{seq}$ to $\mathcal{G}$.

The definition of a pseudo-functor can be found in [CHP04]. To prove this theorem, we need a certain number of technical lemmas that we prove now.

Lemma 5.2.8 :
σ_e is a schedule.

Δ. Let us check that all the conditions are filled.

(i) Let x, $y \in (p_e)$ and z, $t \in (q_e)$ such that $x \leqslant y \sigma_e z \leqslant t$. As φ_f and φ_g are order isomorphisms and that f and g reflect order,

$$\begin{array}{rcl} f^{-1}(\varphi_f(x)) & \leqslant & f^{-1}(\varphi_f(y)) \\ & < & g^{-1}(\varphi_g(z)) \\ & \leqslant & g^{-1}(\varphi_g(t)) \end{array}$$

and thus $x \sigma_e t$.

(ii) Let $x \leqslant p$ be odd, then $\varphi_f(x)$ is of odd height too, thus negative and $f^{-1}(\varphi_f(x))$ is positive. If we have y such that $x \sigma_e y$, then $f^{-1}(\varphi_f(x)) \leqslant g^{-1}(\varphi_g(y))$, thus there is a z such that $f^{-1}(\varphi_f(x)) \prec z \leqslant g^{-1}(\varphi_g(y))$. But as s is innocent, $f(z)\downarrow$ and thus, as f is rigid, lemma **(1.1.5)** implies that $z = f^{-1}(\varphi_f(x+1))$. Therefore $(x+1)\sigma y$.

(iii) Let $x \leqslant q$ be even, then $g^{-1}(\varphi_g(x))$ is positive. Thus, as previously, if $x \sigma_e y$ then $x + 1 \sigma_e y$.

(iv) As minimal elements are in $\mathfrak{D}_g$, $g^{-1}(\varphi_g(1))$ is minimal. Thus, for all $x \in (p)$, $f^{-1}(\varphi_f(x))$ must be superior. □

Lemma 5.2.9 :

Let s be an extremal sequential span, $e \in \mathfrak{D}_g$ if and only if σ_e is right.

Δ. Let $e \in \mathfrak{D}_g$ then let t be the top element of $\lceil e \rceil \cap \mathfrak{D}_f$. Then $t = f^{-1}(\varphi_f(p_e))$ and $e = g^{-1}(\varphi_g(p_e))$ and thus $p_e \sigma_e q_e$ and therefore σ_e is right. The same proof shows that if $e \in \mathfrak{D}_f$ then σ_e is left. □

Lemma 5.2.10 :

Let $e \in S$ not minimal, then $\theta(\pi(e)) = \pi(\theta(e))$.

Δ. Let us suppose that $e \in \mathfrak{D}_g$ and thus, according to lemma (5.2.9), that σ_e is right. Then $\pi(\theta(e)) = (a_e, \pi(b_e), \sigma_e^-)$. Because $e \in \mathfrak{D}_g$, $f(\lceil e \rceil) = f(\lceil \pi(e) \rceil)$ and thus $a_e = a_{\pi(e)}$. Moreover, $g(\lceil \pi(e) \rceil) = \lfloor g(e) \rfloor$ and thus $b_{\pi(e)} = \pi(b_e)$.
Let us now show that $\sigma_e^- = \sigma_{\pi(e)}$. Moreover, as $\lfloor e \rfloor = \lceil \pi(e) \rceil$, $\sigma_{\pi e} = \sigma_{e|p_e \times (q_e - 1)} = \sigma_e^-$. □

Lemma 5.2.11 :

The function θ is injective.

Δ. Let e_1, $e_2 \in S$ such that $\theta(e_1) = \theta(e_2)$. Let us show by induction that $\lceil e_1 \rceil = \lceil e_2 \rceil$.
Let $t_1 \in \lceil e_1 \rceil$ and $t_2 \in \lceil e_2 \rceil$ at the same height, and let us suppose that any two elements lower than them, at the same height, are equal.
Let us first show that t_1 and t_2 are in the same domain. If they are minimal, then they are both in $\mathfrak{D}_g$. If they are not minimal they have, by induction hypothesis, a common direct predecessor, and thus, because of lemma (5.2.4) they must be in the same domain.
If they are in the same domain and, as they have the same strict history, their images are at the same height. Moreover their images are copies of elements in $\lceil a_{e_1} \rceil = \lceil a_{e_2} \rceil$, if t_1 and t_2 are in $\mathfrak{D}_f$, or in $\lceil b_{e_1} \rceil = \lceil b_{e_2} \rceil$, if they are in $\mathfrak{D}_g$. Therefore their images must be copies. But then, because of lemma (5.2.5), we must have $t_1 = t_2$. □

Lemma 5.2.12 :

$G(s)$ is a strategy on $G(A) \multimap G(B)$.

Δ. Let us first check that for all $e \in S$ of height n, $\theta(e) \in (A \multimap B)(n)$. Lemma (5.2.8) already affirms that σ_e is a schedule, moreover, as e is of height n then as their are no internal events and that $\mathfrak{D}_f$ and $\mathfrak{D}_g$ are disjoint, $p_e + q_e = n$. Moreover, $f(\lceil e \rceil)$ is a history and thus its top element $(\_, \_, a_e)$, and therefore a_e, is of height $\#(f(\lceil e \rceil)) = p_e$. It is the same with b_e and thus $\theta(e) \in (A \multimap B)(n)$. Therefore, $G(s) = \{\theta(e) \mid e \in S \text{ of height } n\} \subseteq (A \multimap B)(n)$.
According to lemma (5.2.10), for all $e \in S$, $\pi(\theta(e)) = \theta(\pi(e))$ and thus, $\pi(\theta(e)) \in G(s)$.
Furthermore, if we have x, $y \in S$ even such that $\pi(\theta(x)) = \pi(\theta(y))$, then by lemma (5.2.10), we have $\theta(\pi(x)) = \theta(\pi(y))$. But as θ is injective (lemma (5.2.11)), we have $\pi(x) = \pi(y)$. And thus x and y share a direct predecessor. But then $f(\lceil x \rceil \cup \{y\})$ is down-closed in A and thus a configuration. Therefore, as s is deterministic, $\lceil x \rceil \cup \{y\}$ is a configuration. Therefore, as x and y are even with a common predecessor and s is concrete, if $x \neq y$, they are in conflict, thus they must be equal. We have therefore that $\theta(x) = \theta(y)$.
Finally, let $x \in (A \multimap B)(2n+1)$ such that $\pi(x) \in G(s)(2n)$. Then there exists $e \in S$ such that $\pi(x) = \theta(e)$. Let us suppose that $x := (\sigma, a, b)$ where σ is left. Then q is odd and thus a is even (i.e. negative in $A^\perp$). But then $\lceil a \rceil$ is a negative extension of $f(\lceil e \rceil)$ and thus there exists $X \in \mathcal{C}^o(S)$ such that $f(X) = \lceil a \rceil$. As we have done previously, we can show that $X = \lceil e \rceil \cup \{e'\}$, where $e \prec e'$ and $f(e') = a$.
Then $\pi(\theta(e')) = \theta(\pi(e')) = \theta(e)$. Thus $\theta(a)$ and x have the same direct predecessor. Moreover they are both left, thus they are equal and we have indeed $x \in G(s)(2n+1)$. □

Lemma 5.2.13 :

Let $s_1 = (f_1, S_1, g_1)$ and $s_2 = (f_2, S_2, g_2)$ be sequential relations from A to B and let $\psi : s_1 \Rightarrow s_2$ be a span morphism, then $G(s_1) \subseteq G(s_2)$. Moreover this makes G a functor from $\mathcal{S}pan_{seq}(A, B)$ to $\mathcal{G}(G(A), G(B))$.

Δ. Let $x \in S_1$. Let us show that $\sigma_x \subseteq \sigma_{\psi(x)}$.
First of all $f_1(\lceil x \rceil) = f_2(\lceil \psi(x) \rceil)$ and thus the order isomorphism with (p) is the same (we will call it φ_f). It is the same with φ_g.
Let us now suppose that we have $s\sigma_x t$, i.e.

$$f_1^{-1}(\varphi_f(s)) < g_1^{-1}(\varphi_g(t))$$

Then as $f_1 = f_2 \circ \psi$, $f_1^{-1} = \psi^{-1} \circ f_2^{-1}$ (and similarly for the right side) where ψ^{-1} makes sense as ψ is locally injective. Thus

$$\psi^{-1}(f_2^{-1}(\varphi_f(s))) < \psi^{-1}(g_2^{-1}(\varphi_g(t)))$$

but as ψ preserves the order, we have

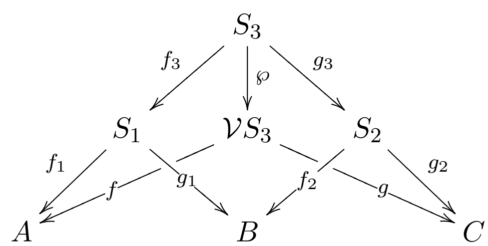
$$f_2^{-1}(\varphi_f(s)) < g_2^{-1}(\varphi_g(t))$$

And thus $s\sigma_{\psi(x)}t$.
As ψ is locally injective it induces a bijection between $\lceil x \rceil$ and $\lceil \psi(x) \rceil$. We therefore have, symmetrically, that $\sigma_{\psi x} \subseteq \sigma_x$ and thus that $\sigma_x = \sigma_{\psi(x)}$. Moreover, as we have already seen, $f_1(\lceil x \rceil) = f_2(\lceil \psi(x) \rceil)$ and thus $a_x = a_{\psi(x)}$ and similarly, $b_x = b_{\psi(x)}$ and thus $\theta(x) = \theta(\psi(x))$.
As s_1 has no internal events, ψ must be total for it to be a span morphism and thus $G(s_1) \subseteq G(s_2)$.
Moreover, as $\mathcal{G}(G(A), G(A))$ is a poset, it is evident that G respects composition. □

Lemma 5.2.14 :

Let s_1 and s_2 be two sequential relations that can be composed. Then $G(s_2 \boxast s_1) = G(s_1) \circ G(s_2)$.

Δ. We have the following diagram



Let $(\sigma \circ \tau, a, c) \in G(s_2) \circ G(s_1)$. Then there exists $b \in B$ such that $(\tau, a, b) \in G(s_1)$ and $(\sigma, b, c) \in G(s_2)$ and thus $e_1 \in S_1$ and $e_2 \in S_2$ such that $\theta(e_1) = (\tau, a, b)$ and $\theta(e_2) = (\sigma, b, c)$.
Let

$$\begin{array}{rll} e_3 := & \{(e, \star) \mid e \in \lceil e_1 \rceil, f_1(e)\downarrow\} & \cup \\ & \{(\star, e') \mid e' \in \lceil e_2 \rceil, g_2(e')\downarrow\} & \cup \\ & \{(e, e') \mid e \in \lceil e_1 \rceil, e' \in \lceil e_2 \rceil, g_1(e) = f_2(e')\} & \end{array}$$

Let us first show that $f_3(e_3) = \lceil e_1 \rceil$. It is clear that $f_3(e_3) \subseteq \lceil e_1 \rceil$. Now, if $e \in \lceil e_1 \rceil$ such that $f_1(e)\downarrow$, then $(e, \star) \in e_3$. But if $g_1(e)\downarrow$, then $g_1(e) \in \lceil b \rceil = f_2(\lceil e_2 \rceil)$ and thus there exists $e' \in \lceil e_2 \rceil$ such that $f_2(e') = g_1(e)$ and therefore $(e, e') \in e_3$.
The symmetric proof shows that $g_3(e_3) = \lceil e_2 \rceil$. Let us now show that $f_3|_{e_3}$ is injective. Let x, $y \in e_3$ such that $f_3(x) = f_3(y)$ (and both defined). Then if $f_1 \circ f_3(x)\downarrow$, both x and y are equal to $(f_3(x), \star) = (f_3(y), \star)$. If $g_1 \circ f_3(x)\downarrow$, then $x := (f_3(x), e)$ and $x := (f_3(y), e')$. But $f_2(e) = g_1 \circ f_3(x) = g_1 \circ f_3(y) = f_2(e')$. As they are both in $\lceil e_2 \rceil$, they are consistent, and thus equal.
Finally if we have x, $y \in e_3$, let us suppose we have $x = (e, \star)$ and $y = (\star, e')$. Then because of innocence both $\lceil e \rceil \cap \mathfrak{D}_{g_1}$ and $\lceil e' \rceil \cap \mathfrak{D}_{f_2}$ must end on an odd event. Thus $g_1(\lceil e \rceil)$ ends on a odd event whereas $f_2(\lceil e' \rceil)$ ends on an even event. Thus they cannot be equal and, as they are both initial segments of a total order,

one must be smaller than the other. Let us suppose $g_1(\lceil e \rceil) \subseteq f_2(\lceil e' \rceil)$. Then there exists $e'' \leqslant e'$ such that $g_1(\lceil e \rceil) = f_2(\lceil e'' \rceil)$ and if we reproduce the construction of e_3 with e and e'', we obtain a subset of e_3 whose projections are configuration, that contain x but not y.
All the other cases are similar (and simpler to treat as we do not have to go look for a pair in between, we already have all the necessary events), and thus we have shown that e_3 is an element of S_3.
Let e_3' be the top non internal event of $\lceil e_3 \rceil$, then it is an element of $\mathcal{V}S_3$. Let us show that $\theta(e_3') = (\sigma \circ \tau, a, c)$. First, as e_3' is the top non internal, we have $f(\lceil e_3' \rceil) = f_1 \circ f_3(\lceil e_3 \rceil) = f_1(\lceil e_1 \rceil) = \lceil a \rceil$ and similarly $g(\lceil e_3' \rceil) = \lceil c \rceil$. Moreover, if we have x, y, z such that $x \tau y \sigma z$, then we have $f_1^{-1}(\varphi_{f_1}(x)) < g_1^{-1}(\varphi_{g_1}(y))$ and $f_2^{-1}(\varphi_{f_2}(y)) < g_2^{-1}(\varphi_{g_2}(z))$ and thus, in e_3,

$$(f_1^{-1}(\varphi_{f_1}(x)), \star) < (g_1^{-1}(\varphi_{g_1}(y)), f_2^{-1}(\varphi_{f_2}(y))) < (\star, g_2^{-1}(\varphi_{g_2}(z)))$$

Thus, we have $x \sigma_{e_3} z$.
Let us now suppose that we have $x \sigma_{e_3} z$, ie $(f_1^{-1}(\varphi_{f_1}(x)), \star) < (\star, g_2^{-1}(\varphi_{g_2}(z)))$. But then lemma (**3.1.3**) implies that there is an element of the form (e, e') between the two. If y is the height of $g_1(e) = f_2(e')$, then we have $x \tau y \sigma z$.
We have proved that $G(s_2) \circ G(s_1) \subseteq G(s_2 \circledast s_1)$.
Let us now consider $(\sigma, a, c) \in G(s_2 \circledast s_1)$. Then there exists $e \in \mathcal{E}S_3$, and thus in S_3, of even height such that $\theta(e) = (\sigma, a, c)$. Let e_1 be the top left element of e and e_2 be the top right.
The top element of $f_1(\lceil e_1 \rceil)$ is a and the top element of $g_2(\lceil e_2 \rceil)$ is c and, if b is the top element of $g_1(\lceil e_1 \rceil)$ then there exists $t_1 \leqslant e_1$ such that $g_1(t_1) = b$. But then there exists $t_2 \leqslant e_2$ such that (t_1, t_2) appears in e and thus $in_2(t_2) = b$.
If t_2 was not the top element in $\lceil e_2 \rceil \cap \mathfrak{D}_{f_2}$ then there would exist $t_2' \in \mathfrak{D}_{in_2}$ such that $t_2 < t_2' \leqslant e_2$ and thus $t_1' \in \mathfrak{D}_{g_1}$ such that $t_1 < t_1' \leqslant e_1$. But that contradicts the fact that b is the top element of $g_1(\lceil s_1 \rceil)$. Thus b is the top element in $f_2(\lceil e_2 \rceil)$ and we have $\theta(e_1) = (\sigma_{e_1}, a, b)$ and $\theta(e_2) = (\sigma_{e_2}, b, c)$.
Let us now show that $\sigma = \sigma_{e_2} \circ \sigma_{e_1}$. First of all, both of these schedules have the same domain, p and the same co-domain, q.
Let x_1 and x_2 such that $x_1 \sigma x_2$. Then there exists y_1, $y_2 \leqslant e$ such that $f(y_1)$ is of height x_1, $g(y_2)$ is of height x_2 and $y_1 \leqslant y_2$.
But, because of lemma (**3.1.3**), $f_3(y_1)$ cannot be the top left element of y_2 and thus there exists a pair (t_1, t_2) in y_2 such that $y_1 \leqslant (t_1, t_2)$. Let x_3 be the height of $g_1(t_1) = f_2(t_2)$. Then $x_1 \sigma_{e_1} x_3 \sigma_{e_2} x_2$. And thus $x_1 (\sigma_{e_2} \circ \sigma_{e_1}) x_2$.
Let us now suppose that we have x_1, x_2 and x_3 such that $x_1 \sigma_{e_1} x_3 \sigma_{e_2} x_2$. Then there exists $t_1 \leqslant t_2 \leqslant e_2$ and $t_3 \leqslant t_4 \leqslant e_2$ such that $f_1(t_1)$ is of height x_1, $g_1(t_2)$ and $f_2(t_3)$ are of height x_3 and $g_2(t_4)$ is of height x_2. But, as $g_1(\lceil e_1 \rceil) = f_2(\lceil e_2 \rceil)$, we must have $g_1(t_2) = f_2(t_3)$. Moreover, t_2 and t_3 must appear in the same pair or else g_1 and f_2 would not be injective on a history, and thus in e we find the pairs $(t_1, \star)$, (t_2, t_3) and $(\star, t_4)$, in this order. Thus $x_1 \sigma x_2$.
Therefore $(\sigma, a, c) \in G(s_2) \circ G(s_1)$. □

Lemma 5.2.15 :

Let A be a filiform concrete event structure, $G(\mathfrak{cc}_A)$ is the copy-cat strategy $\mathfrak{c}_{G(A)}$.

Δ. A look at copycat schedules show that they have the same structure (right, left, left, right, right, ...) that the copycat span in the alternating case. □

Δ(Theorem (5.2.7)). We have proved in lemma (**5.2.13**) that G is a functor on hom categories, in lemma (**5.2.14**) we have proved the existence of an invertible 2-cell between the image of a composition and the composition of the limits and finally in lemma (**5.2.15**) we have proved that there was an invertible 2-cell between $\mathfrak{c}_{G(A)}$ and $G(\mathfrak{cc}_A)$. The naturality of those 2-cells, and the coherence diagram all trivially hold as for all A and B, $\mathcal{G}(A, B)$ is a poset. □

5.3 From strategies to spans

In all that follow, let us suppose that all games are countable, and thus for all A a game, we have an injection $\chi_A : A \to \mathbb{N}$.
Let us now show that we have a functor the other way round and that these two functors are, nearly, reciprocal. To do so, we will first define the functor, then show how it composes with G and finally use this knowledge to prove quite simply that it is a functor.

Definition 5.3.1 (E) **:**
First, let A be a game and let us define $E(A)$ to be the elementary event structure on $\bigsqcup_{n\in\mathbb{N}^*} A(n)$ ordered by $x \leqslant_\pi y$ if and only if it exists $n \in \mathbb{N}$ such that $\pi^n(y) = x$.
Let us now consider A and B two games and s a strategy on $A \multimap B$, let us define

$$S_s := \bigsqcup_{n\in\mathbb{N}^*} s(n)$$

Let $x \leqslant_\pi y$ if and only if it exists $n \in \mathbb{N}$ such that $\pi^n(y) = x$. This defines on order on S_s and thus we can consider S_s as an elementary event structure.
For all left schedules σ let us define $f(\sigma, a, b) = (f(\lfloor(\sigma, a, b)\rfloor), \chi_A(a), a)$ and for all right schedule let us define $g(\sigma, a, b) = (g(\lfloor(\sigma, a, b)\rfloor), \chi_B(b), b)$.
Finally, let $E(s)$ be the span

$$!E(A) \xleftarrow{\;f\;} S_s, \xrightarrow{\;g\;} !E(B)$$

Lemma 5.3.2 :
$E(A)$ is a filiform event structure

∆. Let us show that the order is filiform and finitary. Then $E(A)$ will indeed be an filiform event structure.
First of all, let $x \in A(n)$ and y such that $y \leqslant_\pi x$. By definition, there exists $m \in \mathbb{N}$ such that $y = \pi^m(x)$ and thus $y \in A(n-m)$. Moreover, if there is $z \in A(n-m)$ such that $z \leqslant_\pi x$ then $z = \pi^m(x) = y$. And thus $\lceil x \rceil$ contains at most n elements, i.e. the order is finitary.
Secondly, let $y, z \leqslant_\pi x$. Then there exists n and $m \in \mathbb{N}$ such that $y = \pi^n(x)$ and $z = \pi^m(x)$. Let us suppose that $n \leqslant m$ then $z = \pi^{m-n}(y)$ and thus $z \leqslant y$. Thus the order is filiform. □

Let us now characterise the order $\leqslant_\pi$ on E_s. As many other technical lemmas before, we will admit them, their proof is just rather tedious case study.

Lemma 5.3.3 :
Let σ and τ be two schedules. $(\sigma, a, b) \leqslant_\pi (\tau, a', b')$ if and only if $\sigma \prec \tau$, $a' \leqslant_\pi a$ and $b' \leqslant_\pi b$.

Lemma 5.3.4 :
Let $(\tau, a', b') \in A \multimap B$. For all $a \leqslant_\pi a'$ (respectively $b \leqslant_\pi b'$) there exists a left schedule σ (respectively a right schedule) and $b \leqslant_\pi b'$ (respectively $a \leqslant_\pi a'$) such that $(\sigma, a, b) \leqslant_\pi (\tau, a', b')$. Furthermore, if a (respectively b) is in $A(n)$ then $\sigma = \tau_{\restriction n}$ (respectively $\sigma = \tau^{\restriction n}$).

Lemma 5.3.5 :

If (σ, a, b) and $(\tau, a', b') \leqslant_\pi (\upsilon, a'', b'')$ then the following are equivalent

(i) $(\sigma, a, b) \leqslant_\pi (\tau, a', b')$

(ii) $\sigma \prec \tau$

(iii) $a \leqslant_\pi a'$ and $b \leqslant_\pi b'$

We shall also need the two following results.

Lemma 5.3.6 :

Let $\sigma : p \twoheadrightarrow q$ be a schedule, if σ is right (respectively left) and σ^- is left (respectively right) then $p + q$ is even. Moreover p and q are odd (respectively even).

Δ. let us suppose that $\sigma : p \twoheadrightarrow q$ is left and that $\sigma^- : (p-1) \twoheadrightarrow q$ is right, the proof is similar in the other case. Then $(p-1)$ must be even and q must be odd and thus $p + q$ is even. □

Lemma 5.3.7 :

Let σ and τ be schedules such that $\sigma^- = \tau^-$. Then they must be both left or both right.

Δ. let p and q be such that $\sigma^- : p \to q$ and let us suppose that σ is right and τ is left. Then p must be even and q must be odd. But, according to lemma **(5.3.6)**, $p + q$ must be even. That is absurd. □

Lemma 5.3.8 :

$E(s)$ is a sequential span.

Δ. E_s is a filiform event structure for the same reasons than $E(A)$.
Let us now show that f is well defined. We will show by induction that f is well defined on $\lceil(\sigma, a, b)\rceil$ and that $f(\lceil(\sigma, a, b)\rceil)$ is injected by ε_A unto $\lceil a \rceil$.
Let us suppose this is true for $(\sigma-, a', b') = \pi(\sigma, a, b)$ (if a is minimal the proof is evident). If σ is right, then $a' = a$, f is undefined on (σ, a, b) and $f(\lceil(\sigma, a, b)\rceil) = f(\lceil(\sigma^-, a, b')\rceil)$ that is injected by ε_A unto $\lceil a \rceil$.
If σ is left then $a' = \pi(a)$. As $\lfloor(\sigma, a, b)\rfloor = \lceil \sigma^-, \pi(a), b' \rceil$, $f(\lfloor(\sigma, a, b)\rfloor)$ is injected by ε_A unto $\lceil \pi(a) \rceil = \lfloor a \rfloor$. Thus f is well defined on (σ, a, b) too. Furthermore, $f(\lceil(\sigma, a, b)\rceil) = f(\lceil(\sigma^-, \pi(a), b')\rceil) \cup \{f(\sigma, a, b)\}$, it is injected by ε_A unto $\lceil a \rceil$.
The same proof shows that g is well defined.
Now, as a schedule cannot be both right and left $\mathfrak{D}_f$ and $\mathfrak{D}_g$ are disjoint but as it must be either right or left, there are no internal events. Let us now prove by induction that f reverses parity while g preserves it. If (σ, a, b) is minimal, then it is odd and in $(A \multimap B)(1)$ and thus $\sigma : 0 \twoheadrightarrow 1$. Therefore σ is right and $(\sigma, a, b) \in \mathfrak{D}_g$. But then $g(\sigma, a, b)$ is a copy of $b \in B(1)$ and thus is odd.
Let us now suppose it is not minimal. If it is in the same domain as its direct predecessor, the proof is straight-forward, if it is not, because of lemma **(5.3.6)**, the height of (σ, a, b), $p + q$, is even and if σ is left p is odd. Therefore a is odd and f indeed reverses the parity of (σ, a, b). If σ is right then q is even and thus b is even and g indeed preserves the parity of (σ, a, b).
Let us now show that f is a partial rigid map of event structures. As A is elementary, it is evident that it preserves consistency. Let us show that it also preserves down-closure. Let $X \subseteq E_s$ be a down-closed set and let $(\sigma, a, b) \in X$ be left, and $(P', i', a') \leqslant f(\sigma, a, b) = (P, i, a)$. If they are equal, the proof is finished, if not, $(P', i', a') \in P = f(\lfloor(\sigma, a, b)\rfloor)$ and thus there is $x < (\sigma, a, b)$ such that $f(x) = (P', i', a')$. As X is down-closed, $x \in X$ and thus $f(X)$ is down-closed.
It is evident that f is injective as χ_A is. There remains to show that f is rigid. Let $x \leqslant y$ such that $f(x)$ and $f(y)$ both defined. Then if $x = y$, it is finished, else $f(y) = (f(\lfloor y \rfloor), i, e)$ and thus, as $f(x) \in f(\lfloor y \rfloor)$, $f(x) < f(y)$.

The fact that g is a partial rigid map of event structures is proved the same way.
We have shown so far that $E(s)$ is an alternating, rigid, polarised span. Let us now show it is innocent and negatively saturated.
Let $(\sigma, a, b) \in s$ be right and odd (the case left and odd is the same), where $\sigma : p \twoheadrightarrow q$. Then p is even and thus q is odd. Furthermore, its direct predecessor is $(\sigma^-, a, \pi(b))$. If σ^- was left that would mean that $q-1\sigma^*p$ where $q-1$ is even. Then that would imply that $q\sigma^*p$ but that contradicts that σ is right. Thus σ^- is right too. Thus $f(\pi(\sigma,a,b)) = \pi(b) = \pi(f(\sigma,a,b))$. As the span is alternating we have indeed proved that it is innocent.
The fact that is is negatively saturated is a direct consequence of the fact that all attainable odd move in a strategy are present in the strategy.
Moreover, because s is a strategy there cannot be two events sharing a common odd predecessor, thus all such are in conflict, and we have proved that $E(s)$ is concrete.
As S is elementary, the span is trivially deterministic, as all down-closed subset of S are configurations (without any consideration about their image by f).
It is free of negative coincidence up to copying. Indeed, if we have $\pi(\sigma, a, b) = \pi(\tau, a', b')$ and $f(\sigma, a, b)$ and $f(\tau, a', b')$ are both defined and copies of one another, then σ and τ are both left with the same direct predecessor, thus are equal, and as the schedules are left, $b = b'$. Finally, as the images by f are copies, $a = a'$. The proof in the case where g is defined is the same. ☐

For what follows to work we have to allow span morphism that commute only up to copying, but that does not fundamentally change the bicategory.

Lemma 5.3.9 :

Let s and s' be two strategies on $A \multimap B$ such that there is an inclusion $i : s \subseteq s'$. Then there exists a span morphism $E(i) : E(s) \Rightarrow E(s')$.
Moreover this makes E a functor on the hom-categories.

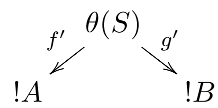
Δ. The inclusion is exactly the span morphism we are looking for, it is then quite easy to see that it is functorial. ☐

Theorem 5.3.10 :

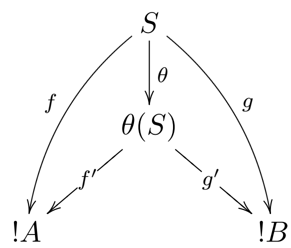
$G \circ E$ is the identity pseudo-functor on $\mathcal{G}$ (seem as an enriched category). $E \circ G$ is the identity on event structures and transforms a sequential span into an isomorphic one.

Δ. Let A be a game, $n \in \mathbb{N}^*$ and $a \in A(n)$. Then $\lceil a \rceil := \{\pi^m(a) \mid 0 \leqslant m < n\}$ and thus a is of height n in $E(A)$. Therefore $G \circ E(A)(n) = A(n)$. Moreover, as $\pi^m(a) \leqslant_\pi \pi(a)$ for all $m > 0$, the immediate predecessor of a in $E(a)$ is $\pi(a)$ and thus $G \circ E(A) = A$.
Let s be a strategy on $A \multimap B$, and $x := (\sigma, a, b) \in s$ (seen as a element of E_s). Let us prove that $\theta(x) = x$. First, $f(\lceil x \rceil) = \lceil a \rceil$ and $g(\lceil x \rceil) = \lceil b \rceil$ and thus $a_x = a$ and $b_x = b$.
Moreover, let us show that $y\sigma z \iff y\sigma_x z$. Let suppose that $y\sigma z$, then $\sigma_{|y} \prec \sigma_{|^z}$ (in fact $(\sigma_{|^z})|_y = \sigma_{|y}$). And thus, as $f^{-1}(\varphi_f(y))$ has for schedule $\sigma_{|y}$, and $g^{-1}(\varphi_g(z))$ has for schedule $\sigma_{|^z}$, according to lemma (**5.3.4**), and that they are both smaller than x, lemma (**5.3.5**) implies that $f^{-1}(\varphi_f(y)) \leqslant_\pi g^{-1}(\varphi_g(z))$ and thus $y\sigma_x z$.
Let us now suppose that $y\sigma_x z$. It implies in particular that $\sigma_{|y} \prec \sigma_{|^z}$ and thus that the domain of $\sigma_{|^z}$ contains y which implies that $y\sigma z$.
Therefore $G \circ E(s) = \theta(E_s) = s$.
Let A be a filiform and concrete event structure, then as all events have a unique height, $A = \bigsqcup_{n \in \mathbb{N}^*} G(A)(n)$. Moreover, as $\pi^n(y) \leqslant_A \pi^{n-1}(y) \leqslant_A \cdot \leqslant_A y$, $x \leqslant_\pi y$ implies $x \leqslant_A y$. Reciprocally, if $x <_A y$ then $x \leqslant_A \pi(y)$ and thus, if we suppose by induction on $\leqslant_A$ that $x \leqslant_\pi \pi(y)$, then, as $\pi(y) \leqslant_\pi y$, $x \leqslant_\pi y$.
Thus A and $E(G(A))$ have the same elements, the same order and are both elementary. They are therefore equal.

Let us now consider $s := (f, S, g)$ a sequential span from A to B. Then $E(G(s))$ is the span



where the order on $\theta(S)$ is determined by π on $A \multimap B$ and the consistency is elementary.
Because of lemma (5.2.9) the following diagram



commutes up to copying. Indeed, let $s \in \mathfrak{D}_f$, the σ_s is left and $\theta(s) = (\sigma_s, in(s), \_)$ and thus $f'(\theta(s))$ is a copy of $f(s)$ and if $s \notin \mathfrak{D}_f$ then $s \in \mathfrak{D}_g$ and thus σ_s is right and therefore $f'(\theta(s))\uparrow$. The commutativity of the other triangle is symmetric.
Let us show that θ is an isomorphism of event structures. First, let us consider $x \leqslant_S y$. Then $x = \pi^n(y)$ for some $n \in \mathbb{N}$ and thus, according to lemma (5.2.10), $\theta(x) = \theta(\pi^n(y)) = \pi^n(\theta(y))$. Therefore $\theta(x) \leqslant_p i\theta(y)$. Moreover as θ is injective, according to lemma (5.2.11),

$$x \leqslant_S y \iff \theta(x) \leqslant_p i\theta(y) \tag{5.1}$$

Let $X \in \mathcal{C}^o(S)$, let us show that $\theta(X)$ is down-closed. Let $x \in X$ and $s \leqslant_\pi \theta(x)$. Then, because θ is surjective on $\theta(S)$, $s = \theta(y)$ for some $y \in S$. But then $y \leqslant_S y$ and as X is down-closed, $y \in X$, thus $s \in \theta(X)$. Moreover as $\theta(S)$ is elementary, it is evident that θ preserves consistency. Finally, as θ is injective, it is injective on configurations. Thus θ is a partial map of event structures.
Moreover, as θ is total and injective, it is injective on configurations and thus a monomorphism. Therefore, it suffices to show it is rigid (which is already done) and surjective on configurations, to show it is a extremal map (equivalent of lemma (3.3.3) in $\mathcal{E}_p$) and thus an isomorphism.
Let X be configuration in $\theta(S)$. As θ is injective and surjective there exist a unique $Y \subseteq S$ such that $\theta(Y) = X$. But then Y is down-closed because of the equivalence (5.1). Furthermore, Y is consistent. □

Theorem 5.3.11 :

E is a pseudo-functor from $\mathcal{G}$ to $\mathcal{S}pan_{seq}$.

Δ. The fact that E is a functor on the hom-cats is shown in lemma (5.3.9)
Let us now show that E respects the composition of strategies up to isomorphism. Let A, B and C be games, s_1 be a strategy on $A \multimap B$ and s_2 be a strategy on $B \multimap C$. From the previous theorem, it follows that

$$G \circ E(s_2 \circ s_1) = (G \circ E)(s_2) \circ (G \circ E)(s_1)$$

and thus

$$G(E(s_2 \circ s_1)) = G(E(s_2) \circ E(s_1))$$

It follows that

$$(E \circ G)(E(s_2 \circ s_1)) = (E \circ G)(E(s_2) \circ E(s_1))$$

and thus that

$$E(s_2 \circ s_1) \simeq E(s_2) \circ E(s_1)$$

Furthermore,

$$E(\mathfrak{c}_A) = E(\mathfrak{c}_{G \circ E(A)}) = E(G(\mathfrak{cc}_{E(A)})) \simeq \mathfrak{cc}_{E(A)}$$

Therefore E is a pseudo-functor. □

5.4 Innocent games

In [HHM07] an other kind of games are described, they are simple games with a monad and a co-monad at the feet based on heaps. But it is quite straight forward to see that a filiform augmentation on an event structure with enough copies, is nothing else than a heap.

The idea is therefore that in we take sequential spans and relax the rigidity condition, we should obtain the second kind of games.

It is true that they restrict the kind of heaps you can have on the left and on the right, but it also happens that innocence as we understand it in our spans, i.e. asking that a negative move in a strategy is always direct successor to the previous positive move, insures that on the right only the player can backtrack, and on the left (because of the dual), only the opponent can backtrack.

It therefore seems that sequential spans without rigidity correspond to the [HHM07] innocent strategies. Moreover, It seems that our composition coincides with the composition of those innocent strategies. But, because of the Kliesli and co-Kliesli constructions, a formal proof would be very technical and thus has not been attempted.

It is true that these are only claims, but they seem true enough and indicate, along with the proof for the simple games, that we have here constructed a very powerful and general notion of strategy.

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Appendix A

Two big diagrams

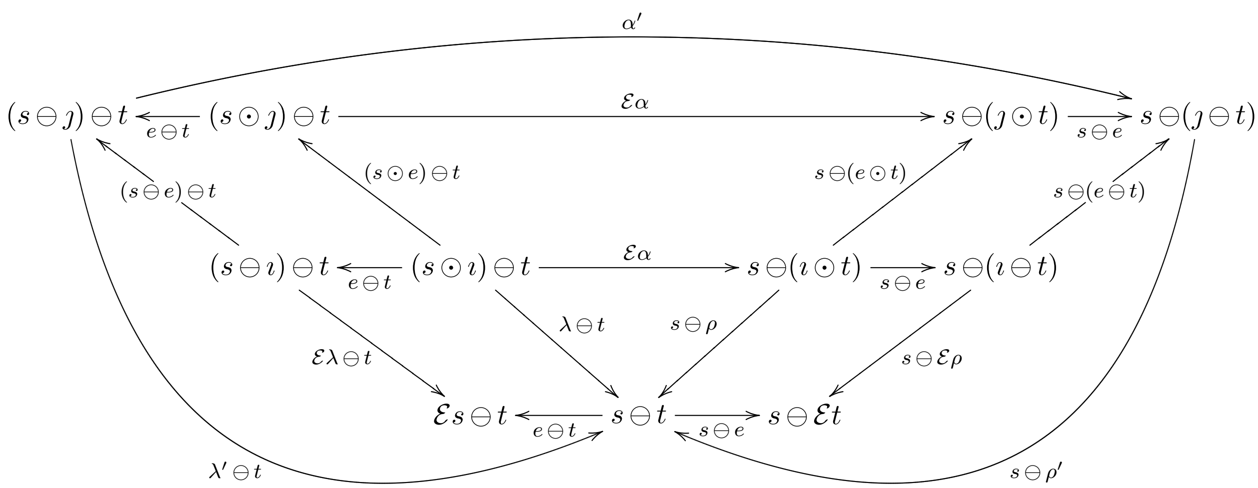

Figure A.1: The triangle

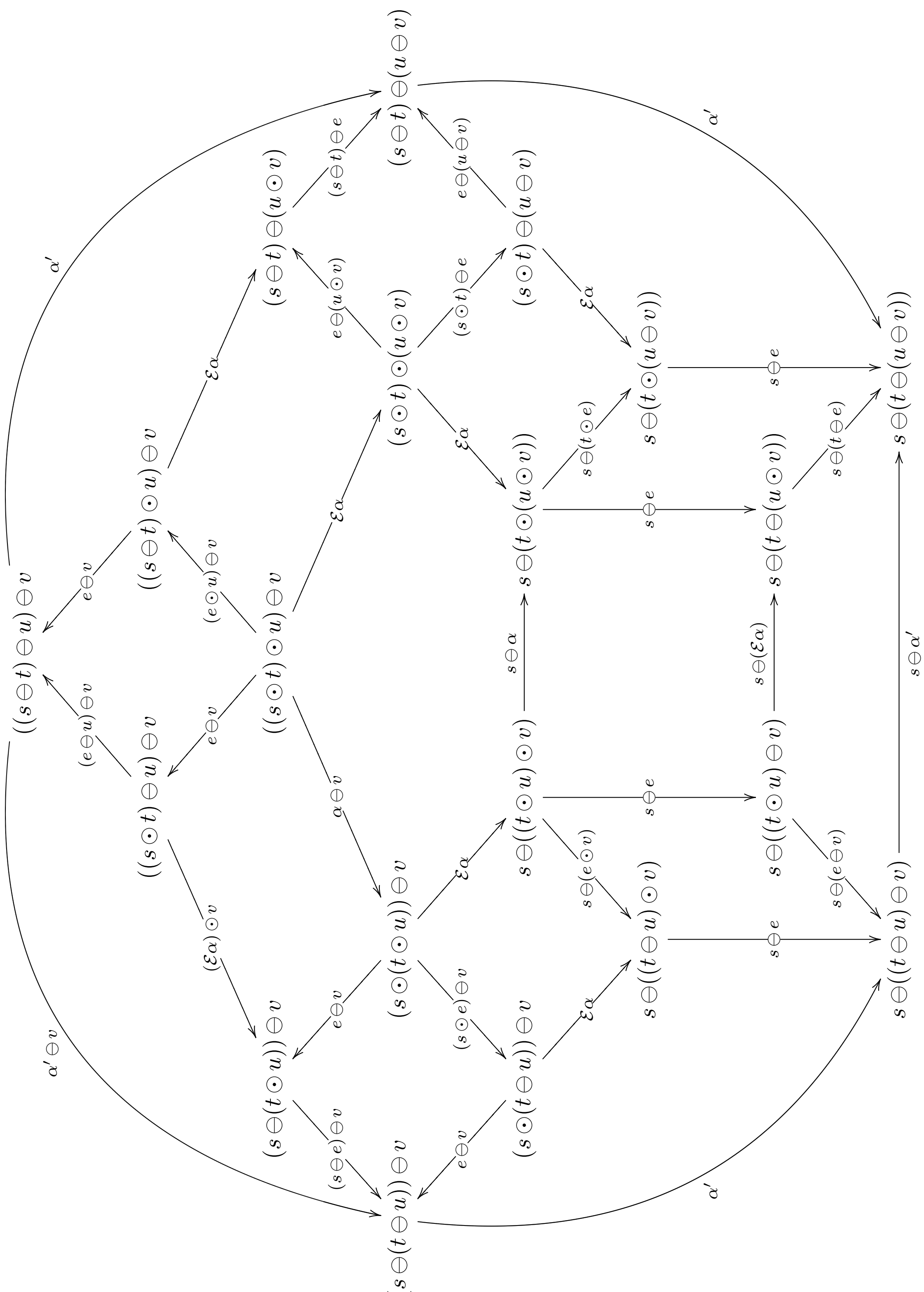

Figure A.2: The pentagon