Note on discrete Linkwitz extension clipping constraints

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Consider the Linkwitz extension filter extending the response from ω_0, Q_0 to ω_1, Q_1:

$$H(s) = \frac{s^2 + \frac{\omega_0}{Q_0}s + \omega_0^2}{s^2 + \frac{\omega_1}{Q_1}s + \omega_1^2}$$

For hardware complexity reasons, it is tempting to implement it in software. This transfer function is a particularly good candidate because its caracteristic frequencies are typically a few hundred hertz, much below audio sampling frequencies.

By applying the bilinear transform (without prewarping) and multiplying both numerator and denominator by $(1+z)^2$, the equivalent digital filter has response

$$H(z) = \frac{(K^2 + K\omega_0/Q_0 + \omega_0^2)z^2 + (2\omega_0^2 - 2K^2)z + (K^2 - K\omega_0/Q_0 + \omega_0^2)}{K^2 + K\omega_0/Q_0 + \omega_0^2)z^2 + (2\omega_0^2 - 2K^2)z + (K^2 - K\omega_0/Q_0 + \omega_0^2)}$$

with $K = 2f_s$. Introducing the caracteristic discrete pulsations $\eta_{0,1} = \omega_{0,1}/K$, we get

$$H(z) = \frac{(1 + \eta_0/Q_0 + \eta_0^2)z^2 + 2(\eta_0 - 1)z + (1 - \eta_0/Q_0 + \eta_0^2)}{(1 + \eta_1/Q_1 + \eta_1^2)z^2 + 2(\eta_1 - 1)z + (1 - \eta_1/Q_1 + \eta_1^2)}$$

Let us call $h(n)$ the impulse response of that filter. If the input sequence is $x \in \ell^\infty(\mathbb{Z})$, then the output y at say $t = 0$ satisfies

$$\begin{aligned} |y| &= \sum_{k=0}^{\infty} x_{-k} h(k) \\ &\leq \sum_{k=0}^{\infty} \|x\|_\infty |h(k)| \\ &\leq \|x\|_\infty \|h\|_1 \end{aligned}$$

where $\|h\|_1$ may *a priori* be infinite. This is tight, for instance with $x_{-k} = \mathrm{sign}(h(k))$ for all $k \in \mathbb{Z}$. Assuming $\|h_1\| < \infty$, this means that the filter, seen as an operator over $\ell^\infty(\mathbb{Z})$, has norm $\|h\|_1$.

In the context of audio processing, this means we need to normalize the filter by the 1-norm of its impulse response to make sure we do not introduce clipping.

We're brought to computing this 1-norm. Fortunately, we know that the (causal) impulse response is given by

$$h(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} H(e^{j\theta}) e^{j\theta n} d\theta$$

The discriminant of $H(z)$'s denominator can be computed to be

$$4\eta_1 \left(2\eta_1^3 - \left(2 + \frac{1}{Q_1^2} \right) \eta_1 - \frac{2}{Q_1} \right) = 4\eta_1 \left(\eta_1 (2\eta_1^2 - (2 + 1/Q_1^2)) - \frac{2}{Q_1} \right)$$

For typical applications, $\eta_1 = \pi \frac{f_1}{f_s} < 1$. This guarantees that the discriminant is negative, so I'll focus on that case from now on.

By partial fraction decomposition, we can write

$$H(z) = G \left(1 + \frac{A}{z - \alpha} + \frac{\overline{A}}{z - \overline{\alpha}} \right)$$

where $\alpha, \overline{\alpha}$ are the complex roots of the denominator, and A can be evaluated as a residue; the residues over the two poles are conjugates because $H(z)$ is a ratio of real polynomials. The norm of α is

$$|\alpha|^2 = \frac{1 - \eta_1/Q_1 + \eta_1^2}{1 + \eta_1/Q_1 + \eta_1^2} < 1$$

Then, we have $h(0) = 1$ and for $n > 0$:

$$|h(n)| = \left| \frac{1}{2\pi} \int_{-\pi}^{\pi} \left(\frac{A}{e^{j\theta} - \alpha} + \frac{\overline{A}}{e^{j\theta} - \overline{\alpha}} \right) e^{j\theta n} d\theta \right|$$

Looking at one half of the integrand, we write

$$\frac{Ae^{j\theta n}}{e^{j\theta} - \alpha} = \frac{Ae^{j\theta(n-1)}}{1 - \alpha e^{-j\theta}}$$

Since $|\alpha| < 1$, we may develop this fraction as a power series, with absolute convergence. It's easy to check we can swap summation and integral, ultimately yielding

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{Ae^{j\theta n}}{e^{j\theta} - \alpha} d\theta = A\alpha^{n-1}.$$

Doing the same computation with the other half, we get in total

$$|h(n)| = |G| \left| 2\Re \left(A\alpha^{n-1} \right) \right|$$

Therefore,

$$\|h\|_1 = |G| \left(1 + \sum_{n=0}^{\infty} |2\Re(A\alpha^n)| \right)$$

In general, this does not have a closed-form expression. However, we are mostly interested in bounding $\|h_1\|$ from above. Call S_N the expression above, where we cut the sum at $n = N$. For terms with $n > N$, bounding by $|\Re(A\alpha^n)| \leq |A||\alpha|^n$, we get that

$$S_N \leq \|h\|_1/|G| \leq S_N + 2|A| \cdot \sum_{n=N+1}^{\infty} |\alpha|^n = S_N + \frac{|2A\alpha^{N+1}|}{1-|\alpha|}$$

We can now extract a procedure to compute $\|h\|_1$ to any given precision: we iterate, keeping S_N and $w = 2A\alpha^N$ in memory. At each step, compute the real part of w and accumulate into S_N. Stop when $|S_N|$ is large enough compared to $|w|/(1-|\alpha|)$. This happens in a reasonable number of steps, since $|w|$ decreases exponentially.